

A CONTRIBUTION TO THE THEORY OF ECONOMIC GROWTH

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I. INTRODUCTION

All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive.¹ A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect.

I wish to argue that something like this is true of the Harrod-Domar model of economic growth. The characteristic and powerful conclusion of the Harrod-Domar line of thought is that even for the long run the economic system is at best balanced on a knife-edge of equilibrium growth. Were the magnitudes of the key parameters — the savings ratio, the capital-output ratio, the rate of increase of the labor force — to slip ever so slightly from dead center, the consequence would be either growing unemployment or prolonged inflation. In Harrod's terms the critical question of balance boils down to a comparison between the natural rate of growth which depends, in the absence of technological change, on the increase of the labor force, and the warranted rate of growth which depends on the saving and investing habits of households and firms.

But this fundamental opposition of warranted and natural rates turns out in the end to flow from the crucial assumption that production takes place under conditions of *fixed proportions*. There is no possibility of substituting labor for capital in production. If this assumption is abandoned, the knife-edge notion of unstable balance seems to go with it. Indeed it is hardly surprising that such a gross

1. Thus transport costs were merely a negligible complication to Ricardian trade theory, but a vital characteristic of reality to von Thünen.

rigidity in one part of the system should entail lack of flexibility in another.

A remarkable characteristic of the Harrod-Domar model is that it consistently studies long-run problems with the usual short-run tools. One usually thinks of the long run as the domain of the neo-classical analysis, the land of the margin. Instead Harrod and Domar talk of the long run in terms of the multiplier, the accelerator, "the" capital coefficient. The bulk of this paper is devoted to a model of long-run growth which accepts all the Harrod-Domar assumptions except that of fixed proportions. Instead I suppose that the single composite commodity is produced by labor and capital under the standard neoclassical conditions. The adaptation of the system to an exogenously given rate of increase of the labor force is worked out in some detail, to see if the Harrod instability appears. The price-wage-interest reactions play an important role in this neoclassical adjustment process, so they are analyzed too. Then some of the other rigid assumptions are relaxed slightly to see what qualitative changes result: neutral technological change is allowed, and an interest-elastic savings schedule. Finally the consequences of certain more "Keynesian" relations and rigidities are briefly considered.

II. A MODEL OF LONG-RUN GROWTH

There is only one commodity, output as a whole, whose rate of production is designated $Y(t)$. Thus we can speak unambiguously of the community's real income. Part of each instant's output is consumed and the rest is saved and invested. The fraction of output saved is a constant s , so that the rate of saving is $sY(t)$. The community's stock of capital $K(t)$ takes the form of an accumulation of the composite commodity. Net investment is then just the rate of increase of this capital stock dK/dt or \dot{K} , so we have the basic identity at every instant of time:

$$(1) \quad \dot{K} = sY.$$

Output is produced with the help of two factors of production, capital and labor, whose rate of input is $L(t)$. Technological possibilities are represented by a production function

$$(2) \quad Y = F(K, L).$$

Output is to be understood as net output after making good the depreciation of capital. About production all we will say at the moment is

that it shows constant returns to scale. Hence the production function is homogeneous of first degree. This amounts to assuming that there is no scarce nonaugmentable resource like land. Constant returns to scale seems the natural assumption to make in a theory of growth. The scarce-land case would lead to decreasing returns to scale in capital and labor and the model would become more Ricardian.²

Inserting (2) in (1) we get

$$(3) \quad \dot{K} = sF(K, L).$$

This is one equation in two unknowns. One way to close the system would be to add a demand-for-labor equation: marginal physical productivity of labor equals real wage rate; and a supply-of-labor equation. The latter could take the general form of making labor supply a function of the real wage, or more classically of putting the real wage equal to a conventional subsistence level. In any case there would be three equations in the three unknowns K , L , real wage.

Instead we proceed more in the spirit of the Harrod model. As a result of exogenous population growth the labor force increases at a constant relative rate n . In the absence of technological change n is Harrod's natural rate of growth. Thus:

$$(4) \quad L(t) = L_0 e^{nt}.$$

In (3) L stands for total employment; in (4) L stands for the available supply of labor. By identifying the two we are assuming that full employment is perpetually maintained. When we insert (4) in (3) to get

$$(5) \quad \dot{K} = sF(K, L_0 e^{nt})$$

we have the basic equation which determines the time path of capital accumulation that must be followed if all available labor is to be employed.

Alternatively (4) can be looked at as a supply curve of labor. It says that the exponentially growing labor force is offered for employment completely inelastically. The labor supply curve is a vertical

2. See, for example, Haavelmo: *A Study in the Theory of Economic Evolution* (Amsterdam, 1954), pp. 9-11. Not all "underdeveloped" countries are areas of land shortage. Ethiopia is a counterexample. One can imagine the theory as applying as long as arable land can be hacked out of the wilderness at essentially constant cost.

line which shifts to the right in time as the labor force grows according to (4). Then the real wage rate adjusts so that all available labor is employed, and the marginal productivity equation determines the wage rate which will actually rule.³

In summary, (5) is a differential equation in the single variable $K(t)$. Its solution gives the only time profile of the community's capital stock which will fully employ the available labor. Once we know the time path of capital stock and that of the labor force, we can compute from the production function the corresponding time path of real output. The marginal productivity equation determines the time path of the real wage rate. There is also involved an assumption of full employment of the available stock of capital. At any point of time the pre-existing stock of capital (the result of previous accumulation) is inelastically supplied. Hence there is a similar marginal productivity equation for capital which determines the real rental per unit of time for the services of capital stock. The process can be viewed in this way: at any moment of time the available labor supply is given by (4) and the available stock of capital is also a datum. Since the real return to factors will adjust to bring about full employment of labor and capital we can use the production function (2) to find the current rate of output. Then the propensity to save tells us how much of net output will be saved and invested. Hence we know the net accumulation of capital during the current period. Added to the already accumulated stock this gives the capital available for the next period, and the whole process can be repeated.

III. POSSIBLE GROWTH PATTERNS

To see if there is always a capital accumulation path consistent with any rate of growth of the labor force, we must study the differential equation (5) for the qualitative nature of its solutions. Naturally without specifying the exact shape of the production function we can't hope to find the exact solution. But certain broad properties are surprisingly easy to isolate, even graphically.

To do so we introduce a new variable $r = \frac{K}{L}$, the ratio of capital to labor. Hence we have $K = rL = rL_0e^{nt}$. Differentiating with respect to time we get

$$\dot{K} = L_0e^{nt}\dot{r} + nrL_0e^{nt}.$$

3. The complete set of three equations consists of (3), (4) and $\frac{\partial F(K,L)}{\partial L} = w$.

Substitute this in (5):

$$(\dot{r} + nr)L_0e^{nt} = sF(K, L_0e^{nt}).$$

But because of constant returns to scale we can divide both variables in F by $L = L_0e^{nt}$ provided we multiply F by the same factor. Thus

$$(\dot{r} + nr)L_0e^{nt} = sL_0e^{nt}F\left(\frac{K}{L_0e^{nt}}, 1\right)$$

and dividing out the common factor we arrive finally at

$$(6) \quad \dot{r} = sF(r, 1) - nr.$$

Here we have a differential equation involving the capital-labor ratio alone.

This fundamental equation can be reached somewhat less formally. Since $r = \frac{K}{L}$, the relative rate of change of r is the difference between the relative rates of change of K and L . That is:

$$\frac{\dot{r}}{r} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}.$$

Now first of all $\frac{\dot{L}}{L} = n$. Secondly $\dot{K} = sF(K, L)$. Making these substitutions:

$$\dot{r} = r \frac{sF(K, L)}{K} - nr.$$

Now divide L out of F as before, note that $\frac{L}{K} = \frac{1}{r}$, and we get (6) again.

The function $F(r, 1)$ appearing in (6) is easy to interpret. It is the total product curve as varying amounts r of capital are employed with one unit of labor. Alternatively it gives output per worker as a function of capital per worker. Thus (6) states that the rate of change of the capital-labor ratio is the difference of two terms, one representing the increment of capital and one the increment of labor.

When $\dot{r} = 0$, the capital-labor ratio is a constant, and the capital stock must be expanding at the same rate as the labor force, namely n .

(The warranted rate of growth, warranted by the appropriate real rate of return to capital, equals the natural rate.) In Figure I, the ray through the origin with slope n represents the function nr . The other curve is the function $sF(r,1)$. It is here drawn to pass through the origin and convex upward: no output unless both inputs are positive, and diminishing marginal productivity of capital, as would be the case, for example, with the Cobb-Douglas function. At the point of intersection $nr = sF(r,1)$ and $\dot{r} = 0$. If the capital-labor ratio r^* should ever be established, it will be maintained, and capital and labor will grow thenceforward in proportion. By constant returns to

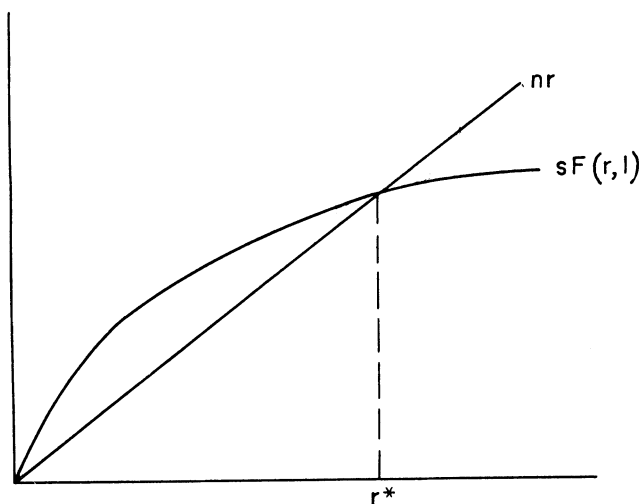


FIGURE I

scale, real output will also grow at the same relative rate n , and output per head of labor force will be constant.

But if $r \neq r^*$, how will the capital-labor ratio develop over time? To the right of the intersection point, when $r > r^*$, $nr > sF(r,1)$ and from (6) we see that r will decrease toward r^* . Conversely if initially $r < r^*$, the graph shows that $nr < sF(r,1)$, $\dot{r} > 0$, and r will increase toward r^* . Thus the equilibrium value r^* is *stable*. Whatever the initial value of the capital-labor ratio, the system will develop *toward* a state of balanced growth at the natural rate. The time path of capital and output will not be exactly exponential except asymptotically.⁴ If the initial capital stock is below the equilibrium ratio,

4. There is an exception to this. If $K = 0$, $r = 0$ and the system can't get started; with no capital there is no output and hence no accumulation. But this

capital and output will grow at a faster pace than the labor force until the equilibrium ratio is approached. If the initial ratio is above the equilibrium value, capital and output will grow more slowly than the labor force. The growth of output is always intermediate between those of labor and capital.

Of course the strong stability shown in Figure I is not inevitable. The steady adjustment of capital and output to a state of balanced growth comes about because of the way I have drawn the productivity curve $F(r,1)$. Many other configurations are a priori possible. For example in Figure II there are three intersection points. Inspec-

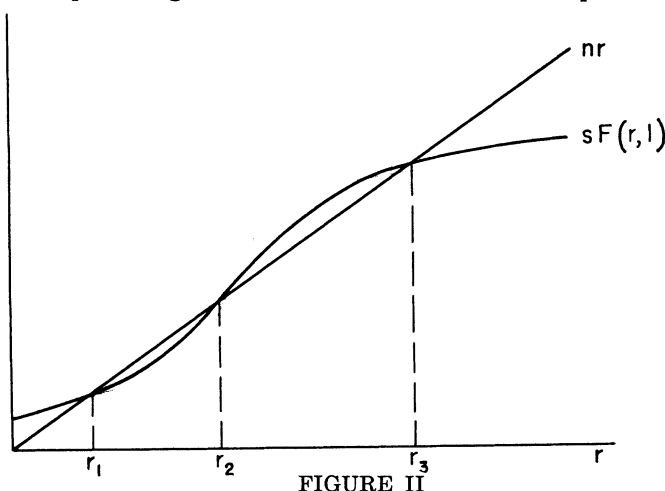


FIGURE II

tion will show that r_1 and r_3 are stable, r_2 is not. Depending on the initially observed capital-labor ratio, the system will develop either to balanced growth at capital-labor ratio r_1 or r_3 . In either case labor supply, capital stock and real output will asymptotically expand at rate n , but around r_1 there is less capital than around r_3 , hence the level of output per head will be lower in the former case than in the latter. The relevant balanced growth equilibrium is at r_1 for an initial ratio anywhere between 0 and r_2 , it is at r_3 for any initial ratio greater than r_2 . The ratio r_2 is itself an equilibrium growth ratio, but an unstable one; any accidental disturbance will be magnified over time. Figure II has been drawn so that production is possible without capital; hence the origin is not an equilibrium "growth" configuration.

Even Figure II does not exhaust the possibilities. It is possible equilibrium is unstable: the slightest windfall capital accumulation will start the system off toward r^* .

that no balanced growth equilibrium might exist.⁵ Any nondecreasing function $F(r,1)$ can be converted into a constant returns to scale production function simply by multiplying it by L ; the reader can construct a wide variety of such curves and examine the resulting solutions to (6). In Figure III are shown two possibilities, together

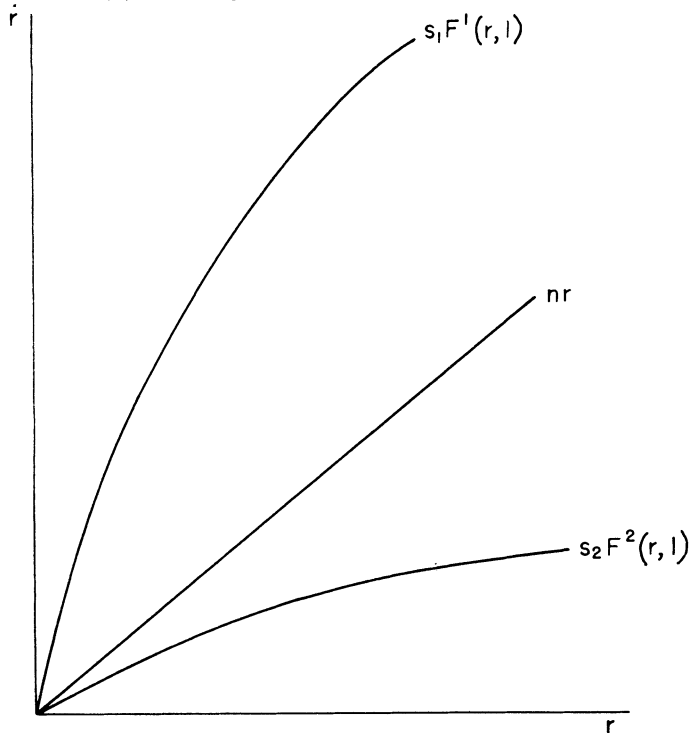


FIGURE III

with a ray nr . Both have diminishing marginal productivity throughout, and one lies wholly above nr while the other lies wholly below.⁶ The first system is so productive and saves so much that perpetual full employment will increase the capital-labor ratio (and also the output per head) beyond all limits; capital and income both increase

5. This seems to contradict a theorem in R. M. Solow and P. A. Samuelson: "Balanced Growth under Constant Returns to Scale," *Econometrica*, XXI (1953), 412-24, but the contradiction is only apparent. It was there assumed that every commodity had positive marginal productivity in the production of each commodity. Here capital cannot be used to produce labor.

6. The equation of the first might be $s_1 F^1(r,1) = nr + \sqrt{r}$, that of the second $s_2 F^2(r,1) = \frac{nr}{r+1}$

more rapidly than the labor supply. The second system is so unproductive that the full employment path leads only to forever diminishing income per capita. Since net investment is always positive and labor supply is increasing, aggregate income can only rise.

The basic conclusion of this analysis is that, when production takes place under the usual neoclassical conditions of variable proportions and constant returns to scale, no simple opposition between natural and warranted rates of growth is possible. There may not be — in fact in the case of the Cobb-Douglas function there never can be — any knife-edge. The system can adjust to any given rate of growth of the labor force, and eventually approach a state of steady proportional expansion.

IV. EXAMPLES

In this section I propose very briefly to work out three examples, three simple choices of the shape of the production function for which it is possible to solve the basic differential equation (6) explicitly.

Example 1: Fixed Proportions. This is the Harrod-Domar case. It takes a units of capital to produce a unit of output; and b units of labor. Thus a is an acceleration coefficient. Of course, a unit of output can be produced with *more* capital and/or labor than this (the isoquants are right-angled corners); the first bottleneck to be reached limits the rate of output. This can be expressed in the form (2) by saying

$$Y = F(K, L) = \min \left(\frac{K}{a}, \frac{L}{b} \right)$$

where “ $\min (. . .)$ ” means the smaller of the numbers in parentheses. The basic differential equation (6) becomes

$$\dot{r} = s \min \left(\frac{r}{a}, \frac{1}{b} \right) - nr.$$

Evidently for very small r we must have $\frac{r}{a} < \frac{1}{b}$, so that in this range

$\dot{r} = \frac{sr}{a} - nr = \left(\frac{s}{a} - n \right) r$. But when $\frac{r}{a} \geq \frac{1}{b}$, i.e., $r \geq \frac{a}{b}$, the equation becomes $\dot{r} = \frac{s}{b} - nr$. It is easier to see how this works graphically. In Figure IV the function $s \min \left(\frac{r}{a}, \frac{1}{b} \right)$ is represented by a

broken line: the ray from the origin with slope $\frac{s}{a}$ until r reaches the value $\frac{a}{b}$, and then a horizontal line at height $\frac{s}{b}$. In the Harrod model $\frac{s}{a}$ is the warranted rate of growth.

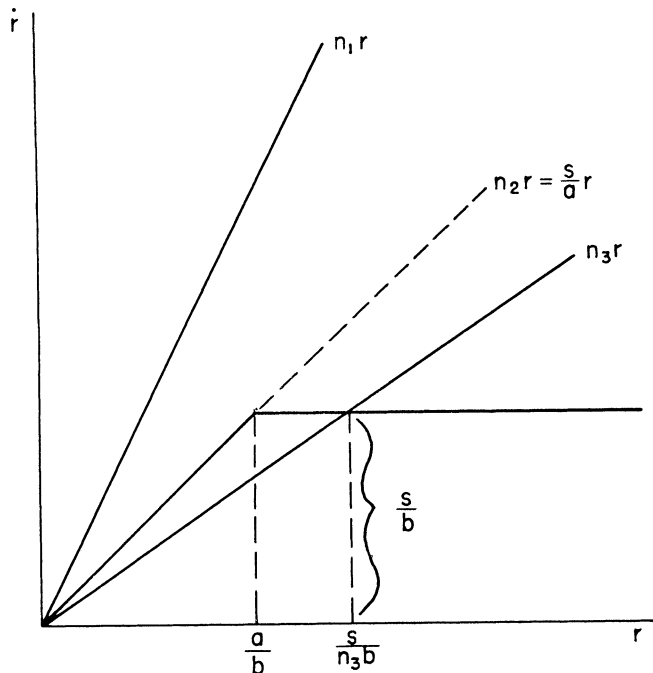


FIGURE IV

There are now three possibilities:

(a) $n_1 > \frac{s}{a}$, the natural rate exceeds the warranted rate. It can be seen from Figure IV that n_1r is always greater than $s \min\left(\frac{r}{a}, \frac{1}{b}\right)$, so that r always decreases. Suppose the initial value of the capital-labor ratio is $r_0 > \frac{a}{b}$, then $\dot{r} = \frac{s}{b} - n_1r$, whose solution is $r = \left(r_0 - \frac{s}{n_1b}\right)e^{-n_1t} + \frac{s}{n_1b}$. Thus r decreases toward $\frac{s}{n_1b}$ which is

in turn less than $\frac{a}{b}$. At an easily calculable point of time t_1 , r reaches

$\frac{a}{b}$. From then on $\dot{r} = \left(\frac{s}{a} - n_1\right)r$, whose solution is $r = \frac{a}{b} e^{\left(\frac{s}{a} - n_1\right)(t-t_1)}$.

Since $\frac{s}{a} < n_1$, r will decrease toward zero. At time t_1 , when $r = \frac{a}{b}$

the labor supply and capital stock are in balance. From then on as the capital-labor ratio decreases labor becomes redundant, and the extent of the redundancy grows. The amount of unemployment can be calculated from the fact that $K = rL_0e^{n_1t}$ remembering that, when capital is the bottleneck factor, output is $\frac{K}{a}$ and employment is $b\frac{K}{a}$.

(b) $n_2 = \frac{s}{a}$, the warranted and natural rates are equal. If initially $r > \frac{a}{b}$ so that labor is the bottleneck, then r decreases to $\frac{a}{b}$ and stays there. If initially $r < \frac{a}{b}$, then r remains constant over time, in a sort of neutral equilibrium. Capital stock and labor supply grow at a common rate n_2 ; whatever *percentage* redundancy of labor there was initially is preserved.

(c) $n_3 < \frac{s}{a}$, the warranted rate exceeds the natural rate. Formally the solution is exactly as in case (a) with n_3 replacing n_1 . There is a stable equilibrium capital output ratio at $r = \frac{s}{n_3b}$. But here capital is redundant as can be seen from the fact that the marginal productivity of capital has fallen to zero. The proportion of the capital stock actually employed in equilibrium growth is $\frac{an_3}{s}$.

But since the capital stock is growing (at a rate asymptotically equal to n_3) the absolute amount of excess capacity is growing, too. This appearance of redundancy independent of any price-wage movements is a consequence of fixed proportions, and lends the Harrod-Domar model its characteristic of rigid balance.

At the very least one can imagine a production function such

that if r exceeds a critical value r_{\max} , the marginal product of capital falls to zero, and if r falls short of another critical value r_{\min} , the marginal product of labor falls to zero. For intermediate capital-labor ratios the isoquants are as usual. Figure IV would begin with a linear portion for $0 \leq r \leq r_{\min}$, then have a phase like Figure I for $r_{\min} \leq r \leq r_{\max}$, then end with a horizontal stretch for $r > r_{\max}$. There would be a whole *zone* of labor-supply growth rates which would lead to an equilibrium like that of Figure I. For values of n below this zone the end result would be redundancy of capital, for values of n above this zone, redundancy of labor. To the extent that in the long run factor proportions are widely variable the intermediate zone of growth rates will be wide.

Example 2: The Cobb-Douglas Function. The properties of the function $Y = K^a L^{1-a}$ are too well known to need comment here. Figure I describes the situation regardless of the choice of the parameters a and n . The marginal productivity of capital rises indefinitely as the capital-labor ratio decreases, so that the curve $sF(r, 1)$ must rise above the ray nr . But since $a < 1$, the curve must eventually cross the ray from above and subsequently remain below. Thus the asymptotic behavior of the system is always balanced growth at the natural rate.

The differential equation (6) is in this case $\dot{r} = sr^a - nr$. It is actually easier to go back to the untransformed equation (5), which now reads

$$(7) \quad \dot{K} = sK^a (L_0 e^{nt})^{1-a}.$$

This can be integrated directly and the solution is:

$$K(t) = \left[K_0^b - \frac{s}{n} L_0^b + \frac{s}{n} L_0^b e^{nb t} \right]^{\frac{1}{b}}$$

where $b = 1 - a$, and K_0 is the initial capital stock. It is easily seen that as t becomes large, $K(t)$ grows essentially like $\left(\frac{s}{n}\right)^{1/b} L_0 e^{nt}$, namely at the same rate of growth as the labor force. The equilibrium value of the capital-labor ratio is $r^* = \left(\frac{s}{n}\right)^{1/b}$. This can be verified by putting $\dot{r} = 0$ in (6). Reasonably enough this equilibrium ratio is larger the higher the savings ratio and the lower the rate of increase of the labor supply.

It is easy enough to work out the time path of real output from the production function itself. Obviously asymptotically Y must

behave like K and L , that is, grow at relative rate n . Real income per head of labor force, Y/L , tends to the value $(s/n)^{a/b}$. Indeed with the Cobb-Douglas function it is always true that $Y/L = (K/L)^a = r^a$. It follows at once that the equilibrium value of K/Y is s/n . But K/Y is the "capital coefficient" in Harrod's terms, say C . Then in the long-run equilibrium growth we will have $C = s/n$ or $n = s/C$: the natural rate equals "the" warranted rate, not as an odd piece of luck but as a consequence of demand-supply adjustments.

Example 3. A whole family of constant-returns-to-scale production functions is given by $Y = (aK^p + L^p)^{1/p}$. It differs from the Cobb-Douglas family in that production is possible with only one factor. But it shares the property that if $p < 1$, the marginal productivity of capital becomes infinitely great as the capital-labor ratio declines toward zero. If $p > 1$, the isoquants have the "wrong" convexity; when $p = 1$, the isoquants are straight lines, perfect substitutability; I will restrict myself to the case of $0 < p < 1$ which gives the usual diminishing marginal returns. Otherwise it is hardly sensible to insist on full employment of both factors.

In particular consider $p = 1/2$ so that the production function becomes

$$Y = (a\sqrt{K} + \sqrt{L})^2 = a^2K + L + 2a\sqrt{KL}.$$

The basic differential equation is

$$(8) \quad \dot{r} = s(a\sqrt{r} + 1)^2 - nr.$$

This can be written:

$$\dot{r} = s[(a^2 - n/s)r + 2a\sqrt{r} + 1] = s(A\sqrt{r} + 1)(B\sqrt{r} + 1)$$

where $A = a - \sqrt{n/s}$ and $B = a + \sqrt{n/s}$. The solution has to be given implicitly:

$$(9) \quad \left(\frac{A\sqrt{r} + 1}{A\sqrt{r_0} + 1} \right)^{1/A} \left(\frac{B\sqrt{r} + 1}{B\sqrt{r_0} + 1} \right)^{-1/B} = e^{\sqrt{ns}t}$$

Once again it is easier to refer to a diagram. There are two possibilities, illustrated in Figure V. The curve $sF(r, 1)$ begins at a height s when $r = 0$. If $sa^2 > n$, there is no balanced growth equilibrium: the capital-labor ratio increases indefinitely and so does real output per head. The system is highly productive and saves-invests enough at full employment to expand very rapidly. If $sa^2 < n$, there is a stable balanced growth equilibrium, which is reached according to

the solution (9). The equilibrium capital-labor ratio can be found by putting $\dot{r} = 0$ in (8); it is $r^* = (1/\sqrt{n/s} - a)^2$. It can be further calculated that the income per head prevailing in the limiting state of growth is $1/(1 - a\sqrt{s/n})^2$. That is, real income per head of labor force will rise to this value if it starts below, or vice versa.

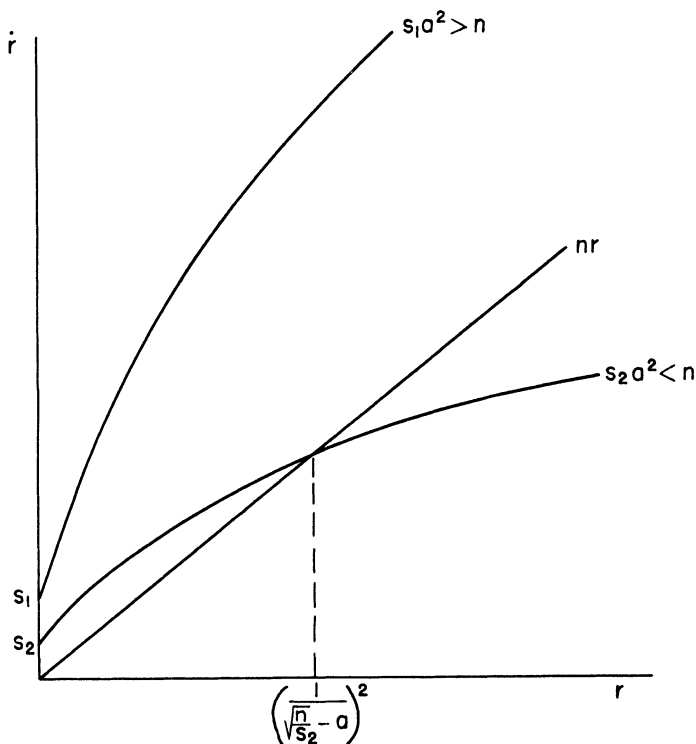


FIGURE V

V. BEHAVIOR OF INTEREST AND WAGE RATES

The growth paths discussed in the previous sections can be looked at in two ways. From one point of view they have no causal significance but simply indicate the course that capital accumulation and real output would have to take if neither unemployment nor excess capacity are to appear. From another point of view, however, we can ask what kind of market behavior will cause the model economy to follow the path of equilibrium growth. In this direction it has already been assumed that both the growing labor force and the

existing capital stock are thrown on the market inelastically, with the real wage and the real rental of capital adjusting instantaneously so as to clear the market. If saving and investment decisions are made independently, however, some additional marginal-efficiency-of-capital conditions have to be satisfied. The purpose of this section is to set out the price-wage-interest behavior appropriate to the growth paths sketched earlier.

There are four prices involved in the system: (1) the selling price of a unit of real output (and since real output serves also as capital this is the transfer price of a unit of capital stock) $p(t)$; (2) the money wage rate $w(t)$; (3) the money rental per unit of time of a unit of capital stock $q(t)$; (4) the rate of interest $i(t)$. One of these we can eliminate immediately. In the real system we are working with there is nothing to determine the absolute price level. Hence we can take $p(t)$, the price of real output, as given. Sometimes it will be convenient to imagine p as constant.

In a competitive economy the real wage and real rental are determined by the traditional marginal-productivity equations:

$$(10) \quad \frac{\partial F}{\partial L} = \frac{w}{p}$$

and

$$(11) \quad \frac{\partial F}{\partial K} = \frac{q}{p}.$$

Note in passing that with constant returns to scale the marginal productivities depend only on the capital-labor ratio r , and not on any scale quantities.⁷

7. In the polar case of pure competition, even if the individual firms have U-shaped average cost curves we can imagine changes in aggregate output taking place solely by the entry and exit of identical optimal-size firms. Then aggregate output is produced at constant cost; and in fact, because of the large number of relatively small firms each producing at approximately constant cost for small variations, we can without substantial error define an aggregate production function which will show constant returns to scale. There will be minor deviations since this aggregate production function is not strictly valid for variations in output smaller than the size of an optimal firm. But this lumpiness can for long-run analysis be treated as negligible.

One naturally thinks of adapting the model to the more general assumption of universal monopolistic competition. But the above device fails. If the industry consists of identical firms in identical large-group tangency equilibria then, subject to the restriction that output changes occur only via changes in the number of firms, one can perhaps define a constant-cost aggregate production function. But now this construct is largely irrelevant, for even if we are willing to overlook

The real rental on capital q/p is an own-rate of interest — it is the return on capital in units of capital stock. An owner of capital can by renting and reinvesting increase his holdings like compound interest at the *variable* instantaneous rate q/p , i.e., like $e^{\int_0^t q/p dt}$. Under conditions of perfect arbitrage there is a well-known close relationship between the money rate of interest and the commodity own-rate, namely

$$(12) \quad i(t) = \frac{q(t)}{p(t)} + \frac{\dot{p}(t)}{p(t)}.$$

If the price level is in fact constant, the own-rate and the interest rate will coincide. If the price level is falling, the own-rate must exceed the interest rate to induce people to hold commodities. That the exact relation is as in (12) can be seen in several ways. For example, the owner of \$1 at time t has two options: he can lend the money for a short space of time, say until $t + h$ and earn approximately $i(t)h$ in interest, or he can buy $1/p$ units of output, earn rentals of $(q/p)h$ and then sell. In the first case he will own $1 + i(t)h$ at the end of the period; in the second case he will have $(q(t)/p(t))h + p(t + h)/p(t)$. In equilibrium these two amounts must be equal

$$1 + i(t)h = \frac{q(t)}{p(t)}h + \frac{p(t + h)}{p(t)}$$

or

$$i(t)h = \frac{q(t)}{p(t)}h + \frac{p(t + h) - p(t)}{p(t)}.$$

Dividing both sides by h and letting h tend to zero we get (12). Thus this condition equalizes the attractiveness of holding wealth in the form of capital stock or loanable funds.

Another way of deriving (12) and gaining some insight into its role in our model is to note that $p(t)$, the transfer price of a unit of capital, must equal the present value of its future stream of net its discontinuity and treat it as differentiable, the partial derivatives of such a function will not be the marginal productivities to which the individual firms respond. Each firm is on the falling branch of its unit cost curve, whereas in the competitive case each firm was actually producing at locally constant costs. The difficult problem remains of introducing monopolistic competition into aggregative models. For example, the value-of-marginal-product equations in the text would have to go over into marginal-revenue-product relations, which in turn would require the explicit presence of demand curves. Much further experimentation is needed here, with greater realism the reward.

rentals. Thus with perfect foresight into future rentals and interest rates:

$$p(t) = \int_t^{\infty} q(u) e^{-\int_t^u r(z) dz} du.$$

Differentiating with respect to time yields (12). Thus within the narrow confines of our model (in particular, absence of risk, a fixed average propensity to save, and no monetary complications) the money rate of interest and the return to holders of capital will stand in just the relation required to induce the community to hold the

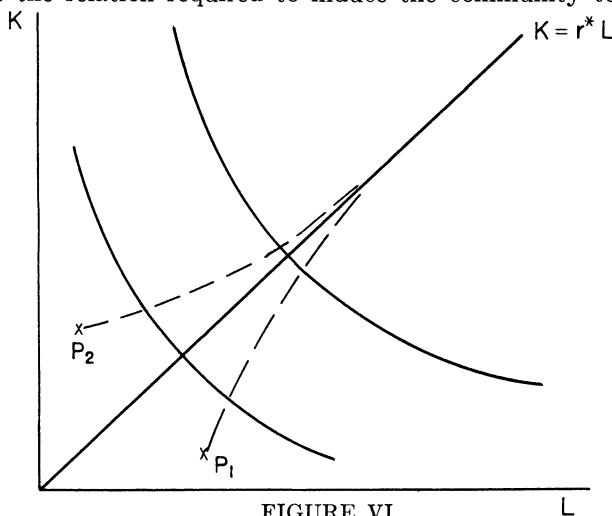


FIGURE VI

capital stock in existence. The absence of risk and uncertainty shows itself particularly in the absence of asset preferences.

Given the absolute price level $p(t)$, equations (10)–(12) determine the other three price variables, whose behavior can thus be calculated once the particular growth path is known.

Before indicating how the calculations would go in the examples of section IV, it is possible to get a general view diagrammatically, particularly when there is a stable balanced growth equilibrium. In Figure VI is drawn the ordinary isoquant map of the production function $F(K, L)$, and some possible kinds of growth paths. A given capital-labor ratio r^* is represented in Figure VI by a ray from the origin, with slope r^* . Suppose there is a stable asymptotic ratio r^* ; then all growth paths issuing from arbitrary initial conditions approach the ray in the limit. Two such paths are shown, issuing from initial

points P_1 and P_2 . Since back in Figure I the approach of r to r^* was monotonic, the paths must look as shown in Figure VI. We see that if the initial capital-labor ratio is higher than the equilibrium value, the ratio falls and vice versa.

Figure VII corresponds to Figure II. There are three "equilibrium" rays, but the inner one is unstable. The inner ray is the dividing line among initial conditions which lead to one of the stable rays and those which lead to the other. All paths, of course, lead upward and to the right, without bending back; K and L always

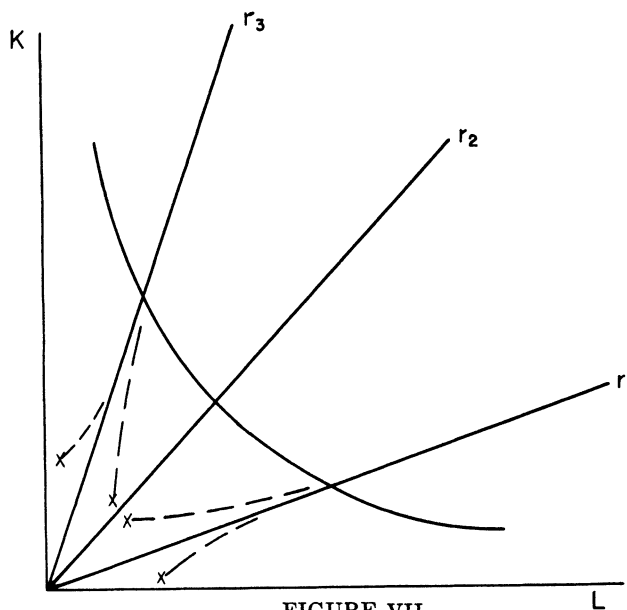


FIGURE VII

increase. The reader can draw a diagram corresponding to Figure III, in which the growth paths pass to steeper and steeper or to flatter and flatter rays, signifying respectively $r \rightarrow \infty$ or $r \rightarrow 0$. Again I remark that K and L and hence Y are all increasing, but if $r \rightarrow 0$, Y/L will decline.

Now because of constant returns to scale we know that along a ray from the origin, the slope of the isoquants is constant. This expresses the fact that marginal products depend only on the factor ratio. But in competition the slope of the isoquant reflects the ratio of the factor prices. Thus to a stable r^* as in Figure VI corresponds an equilibrium ratio w/q . Moreover, if the isoquants have the normal

convexity, it is apparent that as r rises to r^* , the ratio w/q rises to its limiting value, and vice versa if r is falling.

In the unstable case, where r tends to infinity or zero it may be that w/q tends to infinity or zero. If, on the other hand, the isoquants reach the axes with slopes intermediate between the vertical and horizontal, the factor price ratio w/q will tend to a finite limit.

It might also be useful to point out that the slope of the curve $F(r,1)$ is the marginal productivity of capital at the corresponding value of r . Thus the course of the real rental q/p can be traced out in Figures I, II, and III. Remember that in those diagrams $F(r,1)$ has been reduced by the factor s , hence so has the slope of the curve. $F(r,1)$ itself represents Y/L , output per unit of labor, as a function of the capital-labor ratio.

In general if a stable growth path exists, the fall in the real wage or real rental needed to get to it may not be catastrophic at all. If there is an initial shortage of labor (compared with the equilibrium ratio) the real wage will have to fall. The higher the rate of increase of the labor force and the lower the propensity to save, the lower the equilibrium ratio and hence the more the real wage will have to fall. But the fall is not indefinite. I owe to John Chipman the remark that this result directly contradicts Harrod's position⁸ that a perpetually falling rate of interest would be needed to maintain equilibrium.

Catastrophic changes in factor prices do occur in the Harrod-Domar case, but again as a consequence of the special assumption of fixed proportions. I have elsewhere discussed price behavior in the Harrod model⁹ but I there described price level and interest rate and omitted consideration of factor prices. Actually there is little to say. The isoquants in the Harrod case are right-angled corners and this tells the whole story. Referring back to Figure IV, if the observed capital-labor ratio is bigger than a/b , then capital is absolutely redundant, its marginal product is zero, and the whole value of output is imputed to labor. Thus $q = 0$, and $bw = p$, so $w = p/b$. If the observed r is less than a/b labor is absolutely redundant and $w = 0$, so $q = p/a$. If labor and capital should just be in balance, $r = a/b$, then obviously it is not possible to impute any specific fraction of output to labor or capital separately. All we can be sure of is that the total value of a unit of output p will be imputed back to the

8. In his comments on an article by Pilvin, this *Journal*, Nov. 1953, p. 545.

9. R. M. Solow, "A Note on Price Level and Interest Rate in a Growth Model," *Review of Economic Studies*, No. 54 (1953-54), pp. 74-78.

composite dose of a units of capital and b units of labor (both factors are scarce). Hence w and q can have any values subject only to the condition $aq + bw = p$, $aq/p + bw/p = 1$. Thus in Figure IV anywhere but at $r = a/b$ either capital or labor must be redundant, and at a/b factor prices are indeterminate. And it is only in special circumstances that $r = a/b$.

Next consider the Cobb-Douglas case: $Y = K^a L^{1-a}$ and $q/p = a(K/L)^{a-1} = ar^{a-1}$. Hence $w/q = \frac{1-a}{a} r$. The exact time paths

of the real factor prices can be calculated without difficulty from the solution to (7), but are of no special interest. We saw earlier, however, that the limiting capital-labor ratio is $(s/n)^{1/1-a}$. Hence the equilibrium real wage rate is $(1-a)(s/n)^{a/1-a}$, and the equilibrium real rental is an/s . These conclusions are qualitatively just what we should expect. As always with the Cobb-Douglas function the share of labor in real output is constant.

Our third example provides one bit of variety. From $Y = (a\sqrt{K} + \sqrt{L})^2$ we can compute that $\partial Y/\partial L = a\sqrt{\frac{K}{L}} + 1 = a\sqrt{r} + 1$. In the case where a balanced growth equilibrium exists (see end of section IV) $r^* = \left(\frac{1}{\sqrt{n/s} - a}\right)^2$; therefore the limiting real wage is

$w/p = \frac{1}{\sqrt{n/s} - a} + 1 = \frac{1}{1 - a\sqrt{s/n}}$. It was calculated earlier that

in equilibrium growth $Y/L = \left(\frac{1}{1 - a\sqrt{s/n}}\right)^2$. But the relative share

of labor is $(w/p)(L/Y) = 1 - a\sqrt{s/n}$. This is unlike the Cobb-Douglas case, where the relative shares are independent of s and n , depending only on the production function. Here we see that *in equilibrium growth* the relative share of labor is the greater the greater the rate of increase of the labor force and the smaller the propensity to save. In fact as one would expect, the faster the labor force increases the lower is the real wage in the equilibrium state of balanced growth; but the lower real wage still leaves the larger labor force a greater share of real income.

VI. EXTENSIONS

Neutral Technological Change. Perfectly arbitrary changes over time in the production function can be contemplated in principle, but are hardly likely to lead to systematic conclusions. An especially easy kind of technological change is that which simply multiplies the production function by an increasing scale factor. Thus we alter (2) to read

$$(13) \quad Y = A(t)F(K, L).$$

The isoquant map remains unchanged but the output number attached to each isoquant is multiplied by $A(t)$. The way in which the (now ever-changing) equilibrium capital-labor ratio is affected can be seen on a diagram like Figure I by "blowing up" the function $sF(r, 1)$.

The Cobb-Douglas case works out very simply. Take $A(t) = e^{gt}$ and then the basic differential equation becomes

$$\dot{K} = se^{gt}K^a(L_0e^{nt})^{1-a} = sK^aL_0^{1-a}e^{(n(1-a)+g)t},$$

whose solution is

$$K(t) = \left[K_0^b - \frac{bs}{nb+g} L_0^b + \frac{bs}{nb+g} L_0^b e^{(nb+g)t} \right]^{1/b}$$

where again $b = 1 - a$. In the long run the capital stock increases at the relative rate $n + g/b$ (compared with n in the case of no technological change). The eventual rate of increase of real output is $n + ag/b$. This is not only faster than n but (if $a > 1/2$) may even be faster than $n + g$. The reason, of course, is that higher real output means more saving and investment, which compounds the rate of growth still more. Indeed now the capital-labor ratio never reaches an equilibrium value but grows forever. The ever-increasing investment capacity is, of course, not matched by any speeding up of the growth of the labor force. Hence K/L gets bigger, eventually growing at the rate g/b . If the initial capital-labor ratio is very high, it might fall initially, but eventually it turns around and its asymptotic behavior is as described.

Since the capital-labor ratio eventually rises without limit, it follows that the real wage must eventually rise and keep rising. On the other hand, the special property of the Cobb-Douglas function is that the relative share of labor is constant at $1 - a$. The

other essential structural facts follow from what has already been said: for example, since Y eventually grows at rate $n + ag/b$ and K at rate $n + g/b$, the capital coefficient K/Y grows at rate $n + g/b - n - ag/b = g$.

The Supply of Labor. In general one would want to make the supply of labor a function of the real wage rate and time (since the labor force is growing). We have made the special assumption that $L = L_0 e^{nt}$, i.e., that the labor-supply curve is completely inelastic with respect to the real wage and shifts to the right with the size of the labor force. We could generalize this somewhat by assuming that whatever the size of the labor force the proportion offered depends on the real wage. Specifically

$$(14) \quad L = L_0 e^{nt} \left(\frac{w}{p} \right)^h.$$

Another way of describing this assumption is to note that it is a scale blow-up of a constant elasticity curve. In a detailed analysis this particular labor supply pattern would have to be modified at very high real wages, since given the size of the labor force there is an upper limit to the amount of labor that can be supplied, and (14) does not reflect this.

Our old differential equation (6) for the capital-labor ratio now becomes somewhat more complicated. Namely if we make the price level constant, for simplicity:

$$(6a) \quad \dot{r} = sF(r, 1) - nr - h \frac{\dot{w}}{w}.$$

To (6a) we must append the marginal productivity condition (10)

$$\frac{\partial F}{\partial L} = \frac{w}{p}.$$

Since the marginal product of labor depends only on r ,

we can eliminate w .

But generality leads to complications, and instead I turn again to the tractable Cobb-Douglas function. For that case (10) becomes

$$\frac{w}{p} = (1 - a)r^a$$

and hence

$$\frac{\dot{w}}{w} = a \frac{\dot{r}}{r}.$$

After a little manipulation (6a) can be written

$$\dot{r} = (sF(r,1) - nr) \left(1 + \frac{ah}{r}\right)^{-1},$$

which gives some insight into how an elastic labor supply changes things. In the first place, an equilibrium state of balanced growth still exists, when the right-hand side becomes zero, and it is still stable, approached from any initial conditions. Moreover, the equilibrium capital-labor ratio is *unchanged*; since \dot{r} becomes zero exactly where it did before. This will not always happen, of course; it is a consequence of the special supply-of-labor schedule (14). Since r behaves in much the same way so will all those quantities which depend only on r , such as the real wage.

The reader who cares to work out the details can show that over the long run capital stock and real output will grow at the same rate n as the labor force.

If we assume quite generally that $L = G(t, w/p)$ then (6) will take the form

$$(6b) \quad \dot{r} = sF(r,1) - \frac{r}{G} \left(\frac{\partial G}{\partial t} + \dot{w} \frac{\partial G}{\partial \left(\frac{w}{p}\right)} \right).$$

If $\dot{r} = 0$, then $\dot{w} = 0$, and the equilibrium capital-labor ratio is determined by

$$sF(r,1) = \frac{r}{G} \frac{\partial G}{\partial t}.$$

Unless $1/G \partial G/\partial t$ should happen always to equal n , as in the case with (14), the equilibrium capital-labor ratio *will* be affected by the introduction of an elastic labor supply.

Variable Saving Ratio. Up to now, whatever else has been happening in the model there has always been growth of both labor force and capital stock. The growth of the labor force was exogenously given, while growth in the capital stock was inevitable because the savings ratio was taken as an absolute constant. As long as real income was positive, positive net capital formation must result. This rules out the possibility of a Ricardo-Mill stationary state, and suggests the experiment of letting the rate of saving depend on the yield of capital. If savings can fall to zero when income is positive, it becomes possible for net investment to cease and for the capital stock,

at least, to become stationary. There will still be growth of the labor force, however; it would take us too far afield to go wholly classical with a theory of population growth and a fixed supply of land.

The simplest way to let the interest rate or yield on capital influence the volume of savings is to make the fraction of income saved depend on the real return to owners of capital. Thus total savings is $s(q/p)Y$. Under constant returns to scale and competition, the real rental will depend only on the capital-labor ratio, hence we can easily convert the savings ratio into a function of r .

Everyone is familiar with the inconclusive discussions, both abstract and econometrical, as to whether the rate of interest really has any independent effect on the volume of saving, and if so, in what direction. For the purposes of this experiment, however, the natural assumption to make is that the savings ratio depends positively on the yield of capital (and hence inversely on the capital-labor ratio).

For convenience let me skip the step of passing from q/p to r via marginal productivity, and simply write savings as $s(r)Y$. Then the only modification in the theory is that the fundamental equation (6) becomes

$$(6c) \quad \dot{r} = s(r)F(r,1) - nr.$$

The graphical treatment is much the same as before, except that we must allow for the variable factor $s(r)$. It may be that for sufficiently large r , $s(r)$ becomes zero. (This will be the case only if, first, there is a real rental so low that saving stops, and second, if the production function is such that a very high capital-labor ratio will drive the real return down to that critical value. The latter condition is not satisfied by all production functions.) If so, $s(r)F(r,1)$ will be zero for all sufficiently large r . If $F(0,1) = 0$, i.e., if no production is possible without capital, then $s(r)F(r,1)$ must come down to zero again at the origin, no matter how high the savings ratio is. But this is not inevitable either. Figure VII₁ gives a possible picture. As usual r^* , the equilibrium capital-labor ratio, is found by putting $\dot{r} = 0$ in (6c). In Figure VIII the equilibrium is stable and eventually capital and output will grow at the same rate as the labor force.

In general if $s(r)$ does vanish for large r , this eliminates the possibility of a runaway indefinite increase in the capital-labor ratio as in Figure III. The savings ratio *need* not go to zero to do this, but if it should, we are guaranteed that the last intersection with nr is a stable one.

If we compare any particular $s(r)$ with a constant saving ratio, the two curves will cross at the value of r for which $s(r)$ equals the old constant ratio. To the right the new curve will lie below (since I am assuming that $s(r)$ is a decreasing function) and to the left it will lie above the old curve. It is easily seen by example that the equilibrium r^* may be either larger or smaller than it was before. A wide variety of shapes and patterns is possible, but the net effect tends to be stabilizing: when the capital-labor ratio is high, saving is cut down; when it is low, saving is stimulated. There is still no possibility of a stationary state: should r get so high as to choke off

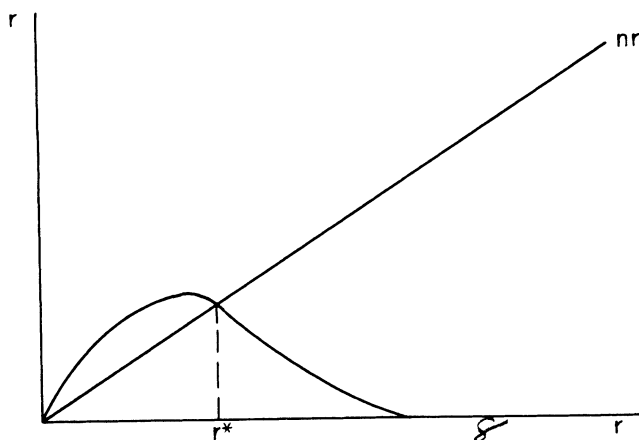


FIGURE VIII

saving and net capital formation, the continual growth of the labor force must eventually reduce it.

Taxation. My colleague, E. C. Brown, points out to me that all the above analysis can be extended to accommodate the effects of a personal income tax. In the simplest case, suppose the state levies a proportional income tax at the rate t . If the revenues are directed wholly into capital formation, the savings-investment identity (1) becomes

$$\dot{K} = s(1 - t)Y + tY = (s(1 - t) + t)Y.$$

That is, the effective savings ratio is *increased* from s to $s + t(1 - s)$. If the proceeds of the tax are directly consumed, the savings ratio is *decreased* from s to $s(1 - t)$. If a fraction v of the tax proceeds is invested and the rest consumed, the savings ratio changes to

$s + (v - s)t$ which is larger or smaller than s according as the state invests a larger or smaller fraction of its income than the private economy. The effects can be traced on diagrams such as Figure I: the curve $sF(r,1)$ is uniformly blown up or contracted and the equilibrium capital-labor ratio is correspondingly shifted. Non-proportional taxes can be incorporated with more difficulty, but would produce more interesting twists in the diagrams. Naturally the presence of an income tax will affect the price-wage relationships in the obvious way.

Variable Population Growth. Instead of treating the relative rate of population increase as a constant, we can more classically make it

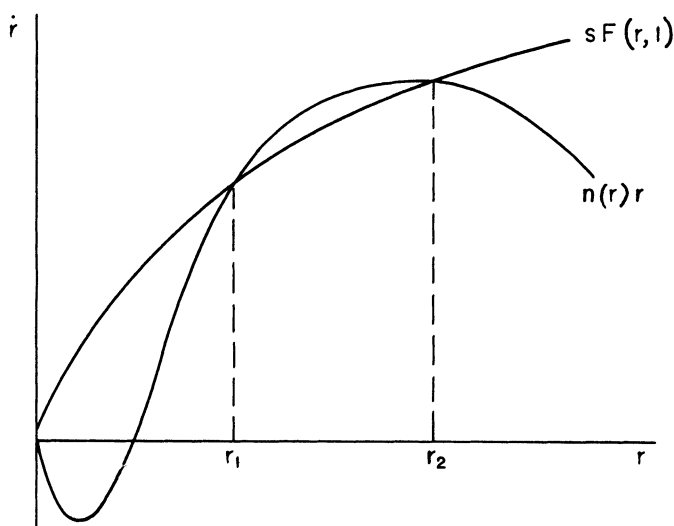


FIGURE IX

an endogenous variable of the system. In particular if we suppose that \dot{L}/L depends only on the level of per capita income or consumption, or for that matter on the real wage rate, the generalization is especially easy to carry out. Since per capita income is given by $Y/L = F(r,1)$ the upshot is that the rate of growth of the labor force becomes $n = n(r)$, a function of the capital-labor ratio alone. The basic differential equation becomes

$$\dot{r} = sF(r,1) - n(r)r.$$

Graphically the only difference is that the ray nr is twisted into a curve, whose shape depends on the exact nature of the dependence

between population growth and real income, and between real income and the capital-labor ratio.

Suppose, for example, that for very low levels of income per head or the real wage population tends to decrease; for higher levels of income it begins to increase; and that for still higher levels of income the rate of population growth levels off and starts to decline. The result may be something like Figure IX. The equilibrium capital-labor ratio r_1 is stable, but r_2 is unstable. The accompanying levels of per capita income can be read off from the shape of $F(r, 1)$. If the initial capital-labor ratio is less than r_2 , the system will of itself tend to return to r_1 . If the initial ratio could somehow be boosted above the critical level r_2 , a self-sustaining process of increasing per capita income would be set off (and population would still be growing). The interesting thing about this case is that it shows how, in the total absence of indivisibilities or of increasing returns, a situation may still arise in which small-scale capital accumulation only leads back to stagnation but a major burst of investment can lift the system into a self-generating expansion of income and capital per head. The reader can work out still other possibilities.

VII. QUALIFICATIONS

Everything above is the neoclassical side of the coin. Most especially it is full employment economics — in the dual aspect of equilibrium condition and frictionless, competitive, causal system. All the difficulties and rigidities which go into modern Keynesian income analysis have been shunted aside. It is not my contention that these problems don't exist, nor that they are of no significance in the long run. My purpose was to examine what might be called the tightrope view of economic growth and to see where more flexible assumptions about production would lead a simple model. Underemployment and excess capacity or their opposites can still be attributed to any of the old causes of deficient or excess aggregate demand, but less readily to any deviation from a narrow "balance."

In this concluding section I want merely to mention some of the more elementary obstacles to full employment and indicate how they impinge on the neoclassical model.¹

Rigid Wages. This assumption about the supply of labor is just the reverse of the one made earlier. The real wage is held at some

1. A much more complete and elegant analysis of these important problems is to be found in a paper by James Tobin in the *Journal of Political Economy*, LXII (1955), 103-15.

arbitrary level $\left(\frac{\bar{w}}{p}\right)$. The level of employment must be such as to keep the marginal product of labor at this level. Since the marginal productivities depend only on the capital-labor ratio, it follows that fixing the real wage fixes r at, say, \bar{r} . Thus $K/L = \bar{r}$. Now there is no point in using r as our variable so we go back to (3) which in view of the last sentence becomes

$$\bar{r} \dot{L} = sF(\bar{r}L, L),$$

or

$$\frac{\dot{L}}{L} = \frac{s}{\bar{r}} F(\bar{r}, 1).$$

This says that *employment* will increase exponentially at the rate $(s/\bar{r})F(\bar{r}, 1)$. If this rate falls short of n , the rate of growth of the labor force, unemployment will develop and increase. If $s/\bar{r}F(\bar{r}, 1) > n$, labor shortage will be the outcome and presumably the real wage will eventually become flexible upward. What this boils down to is that if (\bar{w}/p) corresponds to a capital-labor ratio that would normally tend to decrease ($\dot{r} < 0$), unemployment develops, and vice versa. In the diagrams, $s/\bar{r}F(\bar{r}, 1)$ is just the slope of the ray from the origin to the $sF(r, 1)$ curve at \bar{r} . If this slope is flatter than n , unemployment develops; if steeper, labor shortage develops.

Liquidity Preference. This is much too complicated a subject to be treated carefully here. Moreover the paper by Tobin just mentioned contains a new and penetrating analysis of the dynamics connected with asset preferences. I simply note here, however crudely, the point of contact with the neoclassical model.

Again taking the general price level as constant (which is now an unnatural thing to do), the transactions demand for money will depend on real output Y and the choice between holding cash and holding capital stock will depend on the real rental q/p . With a given quantity of money this provides a relation between Y and q/p or, essentially, between K and L , e.g.,

$$(15) \quad \bar{M} = Q\left(Y, \frac{q}{p}\right) = Q(F(K, L), F_K(K, L))$$

where now K represents capital *in use*. On the earlier assumption of full employment of labor via flexible wages, we can put $L = L_0 e^{nt}$,

and solve (15) for $K(t)$, or employed capital equipment. From $K(t)$ and L we can compute $Y(t)$ and hence total saving $sY(t)$. But this represents net investment (wealth not held as cash must be held as capital). The given initial stock of capital and the flow of investment determine the available capital stock which can be compared with $K(t)$ to measure the excess supply or demand for the services of capital.

In the famous "trap" case where the demand for idle balances becomes infinitely elastic at some positive rate of interest, we have a rigid factor price which can be treated much as rigid wages were treated above. The result will be underutilization of capital if the interest rate becomes rigid somewhere above the level corresponding to the equilibrium capital-labor ratio.

But it is exactly here that the futility of trying to describe this situation in terms of a "real" neoclassical model becomes glaringly evident. Because now one can no longer bypass the direct leverage of monetary factors on real consumption and investment. When the issue is the allocation of asset-holdings between cash and capital stock, the price of the composite commodity becomes an important variable and there is no dodging the need for a monetary dynamics.

Policy Implications. This is hardly the place to discuss the bearing of the previous highly abstract analysis on the practical problems of economic stabilization. I have been deliberately as neo-classical as you can get. Some part of this rubs off on the policy side. It may take deliberate action to maintain full employment. But the multiplicity of routes to full employment, via tax, expenditure, and monetary policies, leaves the nation *some* leeway to choose whether it wants high employment with relatively heavy capital formation, low consumption, rapid growth; or the reverse, or some mixture. I do not mean to suggest that this kind of policy (for example: cheap money and a budget surplus) can be carried on without serious strains. But one of the advantages of this more flexible model of growth is that it provides a theoretical counterpart to these practical possibilities.²

Uncertainty, etc. No credible theory of investment can be built on the assumption of perfect foresight and arbitrage over time. There are only too many reasons why net investment should be at

2. See the paper by Paul A. Samuelson in *Income Stabilization for a Developing Democracy*, ed. Millikan (New Haven, 1953), p. 577. Similar thoughts have been expressed by William Vickrey in his essay in *Post-Keynesian Economics*, ed. Kurihara (New Brunswick, 1954).

times insensitive to current changes in the real return to capital, at other times oversensitive. All these cobwebs and some others have been brushed aside throughout this essay. In the context, this is perhaps justifiable.

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