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Relative Prices as a Function of the Rate of Profit: A Mathematical Note

By

Bertram Schefold, Frankfurt

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1. A New Mathematical Theorem

The reswitching debate has made it obvious that prices are in general complicated functions of the rate of profit in single product systems of the Sraffa[1] type:

$$(1+r)\,Ap + wl = p\text{[2]}.$$

It is well known that productive, indecomposable ("basic") single product systems possess a maximum rate of profit $R>0$ such that the vector of prices expressed in terms of the wage rate $\hat{p}=p/w$ is positive and rises monotonically for $0 \leqslant r < R$, tending to infinity for $r \to R$. The difficulty in analysing such single product systems does not consist in the triviality that prices in terms of the wage rate are an increasing function of the rate of profit, rather it is due to the fact that these prices rise with different "speeds". Relative prices are constant only in one case: when prices are proportional to labour values, i. e. to prices at $r=0$; in general, relative prices deviate from relative labour values for positive rates of profit. (Of course, all these "movements" of prices in function of the rate of profit are purely hypothetical).

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[1] P. Sraffa: Production of Commodities by Means of Commodities, Cambridge 1960.

[2] A is an indecomposable non-negative (n, n)-matrix, l the labour vector, p the vector of relative prices, r the rate of profit, w the wage rate. The coefficient a_i^j of the matrix A denotes the amount of commodity j required to produce one unit of commodity i. The system is supposed to be productive, i. e. $eA \leq e$ where e is the summationvector $e=(1, 1, \ldots, 1)$.

In this article, a mathematical transformation of the price equations is proposed which makes the functional dependence of the price vector on the rate of profit more explicit. This transformation has several economic applications which will be discussed below. The analysis of the origin and the extent of Wicksell effects is improved, it will also be argued that reasonable assumptions about technology do not allow us to exclude the possibility of reswitching. In this respect, this article represents a criticism of Kazuo Sato's attempt to prove that reswitching is empirically irrelevant[3].

Finally, it is proposed to replace the capital-labour ratio which is dimensionally hybrid by the capital-wage ratio which is dimension-free and not subject to perverse Wicksell effetcs.

In order to exhibit the whole generality of the theory we replace the usual single product system by a joint production system

$$(1+r)\, Ap + wl = Bp$$

where B is a non singular output matrix; the joint production system is supposed to be productive, i. e. $e\,A \leqslant e\,B$.

The important properties of the input output matrix of a single product system are summarized in the spectrum of its eigenvalues. However, since the maximum rate of profit has a more direct economic interpretation than the corresponding eigen-value of the input output matrix of a single product system, we consider the roots of the equation det $[B-(1+r)\,A]=0$ instead of looking for the roots of the equation det $(\lambda B-A)=0$. In slight modification of the conventional terminology, we call a root of det $[B-(1+r)\,A)=0$ semi-simple if rank

$$rk\,[B-(1+r)\,A] = n-1.$$

Whether R is a simple root or not: if R is semi-simple there is up to a scalar factor one and only one "eigenvector" q with $q\,[B-(1+R)\,A]=0$. We now assume that det $(B-A)\neq 0$ — a condition which is always fulfilled in productive single product systems. It implies that every vector c of final consumption is producible with non negative activity levels provided $c\,(B-A)^{-1}\geqslant 0$. For the reasons explained below we also assume that det $A\neq 0$. We then have[4]

[3] Kazuo Sato: The neoclassical postulate and the technology frontier in capital theory; The Quarterly Journal of Economics *88* (1974), pp. 353—384.

[4] The non mathematical reader may skip the theorem together with its proof. An alternative proof of the theorem is to be found in B. Schefold:

Theorem 1.1.

Let $R_1, \ldots, R_t$ be the roots of $\det [B-(1+r)A]=0$ with multiplicities $s_1, \ldots, s_t$. The price vector $\hat{p}(r)$ assumes n linearly independent values[5] $\hat{p}(r_1), \ldots, \hat{p}(r_n)$ at any n different rates of profit $r_1, \ldots, r_n$ ($r_i \neq r_j$, $r_i \neq R_j$), if all roots $R_1, \ldots, R_t$ of the equation $\det [B-(1+r)A]=0$ are semi-simple and if $q_i l \neq 0$, $i=1, \ldots, t$, for the associated eigenvectors q_i. Conversely, if one root $\bar{R}$ is not semi-simple or if $\bar{q}_i l=0$ for some $\bar{q}_i$, it follows that $\hat{p}(r_1), \ldots, \hat{p}(r_n)$ are linearly dependent for any $r_1, \ldots, r_n$ ($r_i \neq R_j$) and there will be a vector $\bar{q}$ such that $\bar{q}\hat{p}(r) \equiv 0$ for all r. If $\bar{q}_i$ or $\bar{R}$ is real, there is a real vector $\bar{q}$ with $\bar{q}\hat{p}(r) \equiv 0$ for all r.

Proof: Let s_i be the multiplicity of R_i, R_i semi-simple. We have $\sum_{i=1}^{t} s_i = n$, because $\det A \neq 0$. The roots R_i of $\det [B-(1+r)A]=0$ are the same as those of $\det [I - rA(B-A)^{-1}]$. No root is equal to zero. According to Jordan's theory of Normal Forms[6] (applied to the matrix $A(B-A)^{-1}$), there exist n linearly independent vectors

$$q_{i,1}, \ldots, q_{i,s_i}; \quad i=1, \ldots, t;$$

with

$$q_{i,1} = q_i,$$

$$q_i = R_i q_i A (B-A)^{-1},$$

$$q_{i,\sigma} = R_i q_{i,\sigma} A (B-A)^{-1} - R_i q_{i,\sigma-1}; \quad \sigma = 2, \ldots, s_i.$$

It follows that

$$q_{i,\sigma} [I - rA(B-A)^{-1}] = q_{i,\sigma}\left(1 - \frac{r}{R_i}\right) - r q_{i,\sigma-1}; \quad \sigma=2, \ldots, s_i;$$

and this formula holds for $\sigma=1, \ldots, s_i$, if we define $q_{i,0}=0$ for all i.

Eine Anwendung der Jordanischen Theorie der Normalformen, submitted to Zamp (Zeitschrift für angewandte Mathematik und Physik).

[5] Here and in all of what follows, the price vector is considered as a function of the rate of profit, hence as a *curve* in n-dimensional space in function of the variable r. $\hat{p}(r)$ is said to assume n linearly independent values at the rates of profit $r_1 \ldots, r_n$, if the vectors $\hat{p}(r_1), \ldots, \hat{p}(r_n)$ are linearly independent.

[6] See e. g. W. Gröbner: Matrizenrechnung, Mannheim 1966, pp. 201—205.

With this we get

$$
\begin{aligned}
q_{i,\sigma}(B-A)\,\hat{p}\,(r) &= q_{i,\sigma}(B-A)\,[B-(1+r)\,A]^{-1}\,l \\
&= q_{i,\sigma}\,[I-r\,A\,(B-A)^{-1}]^{-1}\,l \\
&= \frac{R_i}{R_i-r}\,q_{i,\sigma}\,l + \frac{r\,R_i}{R_i-r}\,q_{i,\sigma-1}\,[I-r\,A\,(B-A)^{-1}]^{-1}\,l \\
&= \frac{R_i}{R_i-r}\,q_{i,\sigma}\,l + \frac{r\,R_i}{R_i-r}\;\frac{R_i}{R_i-r}\,q_{i,\sigma-1}\,l \\
&\quad + \frac{(R_i\,r)^2}{(R_i-r)^2}\,q_{i,\sigma-2}\,[I-rA\,(B-A)^{-1}]^{-1}\,l \\
&= \frac{R_i}{R_i-r}\,q_{i,\sigma}\,l + \frac{r\,R_i^2}{(R_i-r)^2}\,q_{i,\sigma-1}\,l + \dots \\
&\quad + \frac{R_i}{R_i-r}\left[\frac{r\,R_i}{R_i-r}\right]^{s_i-1} q_i\,l; \\
&\qquad\qquad \sigma=2,\dots,s_i \quad i=1,\dots,t; \\
q_i\,(B-A)\,\hat{p}\,(r) &= \frac{R_i}{R_i-r}\,q_i\,l.
\end{aligned}
$$

Define (x' denotes the transposed of vector x)

$$
\begin{gathered}
Q = [q'_{1,1},\dots,q'_{1,s_1},\dots,q'_{t,1},\dots,q'_{t,s_t}]', \\
T = Q\,(B-A).
\end{gathered}
$$

The vector $v\,(r)=T\hat{p}\,(r)$ assumes in n points $r_1,\dots,r_n$ ($r_i\neq r_j$, $r_i\neq R_j$) n linearly independent values, if and only if $q_i\,l\neq 0$; $i=1,\dots,t$. The necessity of this latter condition is obvious, for $q_i\,(B-A)\,\hat{p}\,(r)\equiv 0$ if $q_i\,l=0$. To verify that $q_i\,l\neq 0$ is sufficient, consider the matrix

$$U=[u\,(r_1),\dots,u\,(r_n)]$$

where $u\,(r)=\det\,[B-(1+r)\,A]\;v\,(r)$. The components of the vector $u\,(r)$ are denoted by $u_{i,\sigma}$ where $i=1,\dots,t$; $\sigma=1,\dots,s_i$. We have

$$
\begin{aligned}
u_{i,\sigma} = [R_i\,(R_i-r)^{s_i-1}\,q_{i,\sigma}\,l + \dots + r^{\sigma-1}\,R_i^{\sigma}\,(R_i-r)^{s_i-\sigma}\,q_i\,l]\,\times \\
\times \prod_{j\neq i}\,(R_j-r)^{s_j}.
\end{aligned}
$$

We show that U is not singular by showing that $x=0$ for any vector x such that $xU=0$. Since each row of U consists of the values at n points of a polynomial of degree $n-1$, and since the values at n points determine a polynomial of degree $n-1$ fully, $xu\,(r)=0$ for $r=r_i$, $i=1,\dots,n$, implies $xu\,(r)\equiv 0$ for all r. Denoting the components of x in the same way as those of u we find at once that x_{i,s_i} $(i=1,\dots,t)$ must be zero because $u_{i,s_i}(R_i)\neq 0$ while $u_{j,\sigma}(R_i)=0$ for $(j,\sigma)\neq(i,s_i)$. If $s_i>1$, we must have

$$\sum_{(j,\sigma)\neq(i,s_i)} x_{j,\sigma}\frac{u_{j,\sigma}}{R_i-r}\equiv 0.$$

Hence $x_{i,s_i-1}=0$ since $\lim\limits_{r\to R_i}\frac{u_{i,s_i-1}}{R_i-r}\neq 0$ while $\lim\limits_{r\to R_i}\frac{u_{j,\sigma}}{R_i-r}=0$ for $(j,\sigma)\neq(i,s_i)$ and $(j,\sigma)\neq(i,s_i-1)$.

Continuing the induction one obtains that $x=0$. Thus U is non-singular and

$$\hat{p}(r)=T^{-1}v(r)$$

assumes n linearly independent values at n different points, if the R_i are semi-simple, and if $q_i l\neq 0$, $i=1,\dots,t$. The necessity of R_i being semi-simple remains to be shown. Suppose R is a multiple root of $\det[B-(1+R)A]=0$ and $rk\ [B-(1+R)A]<n-1$. There are then two linearly independent q_1, q_2 with

$$q_i[B-(1+R)A]=0;\quad i=1,2;\quad q_i l\neq 0.$$

We get as above

$$q_i(B-A)(B-(1+r)A]^{-1}l=q_i[I-rA(B-A)^{-1}]^{-1}l=\frac{R}{R-r}q_il;\ i=1,2.$$

If $q_2l=0$, define $\bar{q}=q_2(B-A)$. Otherwise, we have

$$(q_1-\lambda q_2)(B-A)\hat{p}(r)\equiv 0,\ \lambda=\frac{q_1l}{q_2l},$$

identically in r with $\bar{q}=(q_1-\lambda q_2)(B-A)\neq 0$, which is impossible, if $\hat{p}(r)$ assumes n linearly independent values in any n points. $\bar{q}$ is a real vector, if R is real.

q. e. d.

It remains to discuss how the statements of the theorem are affected if A is a singular matrix.

There is only one relevant economic reason why A could be singular: if the system contains pure consumption goods, entire columns of A will be zero. More generally, if the rank of A is $n-z$, there will be z vectors $q_1,\dots,q_z$ with $q_iA=0$ so that

$$q_i(B-A)[B-(1+r)A]^{-1}l=q_i[I-rA(B-A)^{-1}]^{-1}l=q_il.$$

It follows, as in the proof above, that there will be a vector q

with $\bar{q}\hat{p}(r) \equiv 0$ if $z \geqslant 2$ and/or $q_i l = 0$ for some i. Hence, systems with rank $A < n-1$ are not regular in the sense of the following section.

2. Values and Prices: The Rule and the Exception

If a Sraffa joint production system with det $A \neq 0$, det $(B-A) \neq 0$ has only semi-simple characteristic roots and if none of its eigenvectors is orthogonal to its labour vector, we shall call it *regular*[7]. The theorem then says that the price vector of regular Sraffa systems with n commodities and n industries varies in such a way with the rate of profit that it assumes n linearly independent values at any n different levels of the rate of profit. This means that the price vector of a regular Sraffa system is not only not constant, but its variations in function of the rate of profit result in a complicated twisted curve such that the n price vectors belonging to n different levels of the rate of profit r_i span a $(n-1)$-dimensional hyperplane which never contains the origin (provided $r_i \neq R_j$).

Irregular Sraffa systems on the other hand (i. e. those which have a characteristic equation with a multiple root and/or an eigenvector which is orthogonal to the labour vector) are such that the n price vectors taken at n different levels of the rate of profit can *never* be linearly independent. We must ask ourselves: are regular systems the exception or the rule? What is the economic interpretation of the theorem? What economic interpretation do the exceptions have?

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[7] We note the following corollary:

Corollary: The systems considered above are regular if and only if the vectors $l, Al, \ldots, A^{n-1}l$ are linearly independent.

Proof: If $q_i l = 0$ for some eigenvector q_i of A, the matrix $F = (l, Al, \ldots$ $\ldots, A^{n-1}l)$ is not regular, since $q_i F = 0$. If $q_i l \neq 0$, $i = 1, \ldots, t$, and if R is not a semi-simple root of the characteristic equation det $[I-(1+r)A] = 0$, there are at least two eigenvectors q_1, q_2 associated with R such that $\bar{q}F = 0$ where $\bar{q} = q_1 + \mu q_2$, $\mu = -(q_1 l / q_2 l)$. Conversely, if the system is regular, and if m is the maximum number such that the vectors $l, Al, \ldots$ $\ldots, A^m l$ are linearly independent, suppose that the subspace $R^{m+1} \subset R^n$ spanned by the vectors $l, Al, \ldots, A^m l$ had dimension $m+1 < n$.

Since the matrix $I-(1+r)A$ maps R^{m+1} onto itself, with det $[I-(1+r)A] \neq 0$, this would imply that $p(r) = [I-(1+r)A]^{-1} l \in R^{m+1}$ for *all* $r \neq R_i$, $i = 1, \ldots, t$, which contradicts theorem 1.1 above.

(This corollary is used in B. Schefold: "Nachworte", section 6, in P. Sraffa, "Warenproduktion mittels Waren", Berlin 1976, pp. 131—226.)

First of all, it is easy to see that the regular systems are the rule from a mathematical point of view because even multiple roots are mathematically exceptional and because an eigenvector will only by coincidence be orthogonal to the labour vector. To put it in more precise terms: One can easily prove that the set of irregular Sraffa systems with (n, n)-input-output-matrices is of measure zero in the set of all Sraffa systems with the same number of commodities and industries.

But this observation taken by itself does not mean much. The set of all semi-positive decomposable (n, n)-matrices is also of measure zero in the set of all semi-positive (n, n)-matrices, and yet it is quite clear that the analysis of the "exceptional" decomposable matrices is of greatest economic interest, although they are more difficult to handle than indecomposable matrices. There is an excellent economic reason why decomposable systems are important: pure consumption goods and other non basics exist; therefore decomposable systems exist.

I should like to argue that matters are quite different with irregular systems. I believe that there is no economic reason why real systems should not be regular or why irregular systems should exist in reality; irregularity is only a fluke, or, at best, an approximation. But there is one kind of irregular system which is very useful to the economists because it provides a simple abstraction from some of the more complicated properties of regular systems. *Die Ausnahme bestätigt die Regel* — the exception confirms the rule. By considering the exception we learn why regular systems are the normal case. The extreme form of an irregular system is one where relative prices are constant, i. e. equal to relative labour values.

Relative prices will be constant and equal to relative labour values if (supposing we are dealing with single product systems)

$$(1+R)\,Al = l$$

i. e. if the labour vector happens to be a right hand eigenvector of the input output matrix. All left hand eigenvectors not associated with the characteristic root R will then be orthogonal to l, because for any left hand eigenvector $\bar{q}$ associated with an eigenvalue $\bar{R} \neq R$, we get

$$(1+\bar{R})\,\bar{q}\,Al = \bar{q}l = (1+R)\,\bar{q}\,Al$$

therefore

$$\bar{q}\,Al = \bar{q}l = 0.$$

It follows from the proofs of our theorem above that

$$q_{i,\sigma_i}(B-A)\,\hat{p}\,(r)\equiv 0$$

for all q_{i,σ_i} except the eigenvector associated with $1+R$. This means that $\hat{p}\,(R)$ is a scalar multiple of a constant vector which is a somewhat roundabout proof that relative prices are equal to relative labour values in this case. (The direct proof is obvious.)

Conversely: if prices are equal to values, the price equation must hold at all rates of profit with constant prices. Putting formally $r=-1$, we get $w\,(-1)\,l=p$. Inserting this into the price equation, we obtain $(1+r)\,Aw\,(-1)\,l+w\,(r)\,l=w\,(-1)\,l$, which implies that l is an eigenvector of A. Because of $l>0$, this eigenvector must belong to the eigenvalue corresponding to the maximum rate of profit. Thus, the condition $(1+R)\,Al=l$ is necessary and sufficient for values being equal to prices in a productive and indecomposable single product system.

If relative prices are equal to relative labour values and if absolute prices are expressed in terms of a commodity, it does not matter which of the commodities is taken as a numéraire; the wage rate will always be related to the rate of profit by a simple linear relationship, because the price in terms of the commodity numéraire of total income Y and of capital K is then constant so that the sum of wages W and profits P can be written as

$$Y=W+P=W+rK$$

therefore

$$W=Y-rK.$$

We verify that if

$$(1+R)\,Al=l$$

the price equation

$$(1+r)\,Ap+wl=p$$

is fulfilled for

$$p=(1+1/R)\,l$$

and

$$w=1-r/R$$

with p being the sum of direct labour l and indirect labour $(1/R)\,l$. By assuming that prices are equal to labour values, one abstracts from the problem of relative prices and focuses attention on the problem of determining the relation between the distribution of income (wages or a share in income) and the rate of profit: An increase in the rate of profit engenders a proportionate diminution of the wage rate and, since the level of employment is given, of wages.

If prices are not equal to values, we get the same simple relationship between wages and the rate of profit only if prices are expressed in terms of a suitable average of commodities: Mr. Sraffa's "Standard Commodity". There is a unique positive eigenvector q associated with the maximum rate of profit in an indecomposable ("basic") single product system, such that

$$(1+R)\,qA = q.$$

Hence

$$q\,(I-A)\,p = q\,(rAp + wl) = (r/R)\,q\,(I-A)\,p + wql$$

i. e.

$$1 = w + r/R$$

as above if prices are expressed in terms of the "Standard Commodity" $[q\,(I-A)\,p=1]$ and if the eigenvector q is normalized so that $ql=1$. In other words, if prices are expressed in terms of a "standard commodity", we can abstract from the complications arising from relative prices and obtain in the general case a linear relationship between wages and the rate of profit which is of the same form as that holding for all commodity price standards in systems where prices equal values. (Total income, however, is not constant in terms of the standard commodity, unless the economy itself is in standard proportions.)

This fact has often been commented upon. What we now learn is that in between the most extreme and most "exceptional" case where prices are equal to values and the "regular" case where prices vary along a twisted curve in function of the rate of profit, there are intermediate cases where (possibly real) vector $\bar{q}$ exists, such that $\bar{q}\hat{p}\,(r) \equiv 0$ so that at least one of the components of the price vector is a linear function with constant coefficients of the other components of the price vector. In formulas: $\bar{q}\hat{p}\,(r) \equiv 0$ implies

$$\hat{p}_i\,(r) \;=\; 1/\bar{q}_i\,(\bar{q}_1\hat{p}_1 + \ldots + \bar{q}_{i-1}\hat{p}_{i-1} + \bar{q}_{i+1}\hat{p}_{i+1} + \ldots + \bar{q}_n\hat{p}_n)$$

where $\bar{q}_i \neq 0$.

But the price of one commodity can only be a linear function of the prices of the other commodities if there is an inner technical relationship between the processes of production of the commodities. (If prices equal values, the technical relationship must take the form $(1+R)\,Al=l$). I cannot think of any economic reason why such a relationship should exist. This in general confirms that irregular systems are exceptional and interesting only in so far as they allow to abstract from the complications of regular systems[8].

Going a little further, we may conlcude that the normal case is one where all roots of the system are simple and not only semi-simple because the set of input-output matrices with multiple roots is of measure zero in the set of all input-output matrices of given order. In the case of simple roots the proof of our theorem yields that the vector of prices in terms of the wage rate $\hat{p}$ may be written as

$$\hat{p}\,(r) = S \begin{bmatrix} \frac{R_1}{R_1 - r} q_1 l \\ \vdots \\ \frac{R_n}{R_n - r} q_n l \end{bmatrix}$$

where $S = T^{-1}$ is a non singular matrix, where the $R_1, \ldots, R_n$ are the n distinct roots of the characteristic equation of the system, and where $q_i\, l \neq 0$, $i = 1, \ldots, n$. This formula shows the functional dependence of the price vector on the rate of profit in a (from the mathematical point of view) simple and explicit form: The n-dimensional complex space C^n is mapped onto itself by the matrix T in such a way that each component of $\hat{p}\,(r)$ is a simple hyperbola in function of the parameter r with a singularity at $r = R_i$.

3. Uniqueness of the System Yielding a Given Vector of Prices

Regular systems are important because the complicated behaviour of their prices implies that the technique does not only determine prices in function of the rate of profit, but that the converse is also true: If the price vector is given at $n+1$ different levels of the rate of profit, there is essentially only one technique which is compatible with those prices. The result derives from the following three theorems.

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[8] Systems where a labour theory of value holds have been discussed innumerable times. Perhaps it is instructive to give an example of a single product system where one of the roots is multiple. For the input matrix

$$A = \begin{Bmatrix} 1/3 & 1/4 & 1/4 \\ 1/4 & 1/3 & 1/4 \\ 1/4 & 1/4 & 1/3 \end{Bmatrix}$$

we obtain in det $[I - (1+r)\,A] = 0$ a simple root (maximum rate of profit) if $r = 1/5$ and a double root for $r = 5/4$. This double root is not semi-simple.

Theorem 3.1:

Let two (n, n) joint production systems be given with input matrices A, F, output matrices B, G and labour vectors l, m respectively. If and only if the vector of relative prices in terms of the wage rate is the same for both systems at every level of the rate of profit, the two systems are related by the equations

$$G = MB + Y$$
$$F = MA + Y$$
$$m = Ml$$

where M is a non singular (n, n)-matrix and where $Y \hat{p}\,(r) \equiv 0$ for all r.

Proof: Define $M = (G - F)\,(B - A)^{-1}$ and $Y = F - MA$. From

$$l = [B - (1+r)\,A]\,\hat{p}$$
$$Ml = M\,[B - (1+r)\,A]\,\hat{p}$$

and

$$m = [G - (1+r)\,F]\,\hat{p}$$

we obtain

$$m = M\,[B - (1+r)\,A]\,\hat{p} - rY\hat{p},$$

therefore

$$m = Ml - rY\hat{p}\,(r).$$

Hence $m = Ml$ and $Y\hat{p}\,(r) \equiv 0$. The converse is obvious.

q. e. d.

Theorem 3.2:

If prices of two joint production systems coincide at $n+1$ different levels of the rate of profit, they coincide everywhere.

Proof: If the equation (using the notation of the previous theorem)

$$[G - (1+r)\,F]\,[B - (1+r)\,A]^{-1} = m$$

holds in $n+1$ points, we also have in $n+1$ points

$$(G - F - rF)\,(B - A - rA)_{\mathrm{Ad}}\,l = \det\,[B - (1+r)\,A]\,m$$

where the subscript *Ad* means the adjoint of the corresponding matrix. On both sides of the equation we have a vector of polynomials of degree n. Since the polynomials coincide in $n+1$ points, they coincide everywhere.

q. e. d.

Since the output matrices of two single product systems of the same order are trivially identical (therefore $M = I$) and since $Y\hat{p}(r) \equiv 0$ implies $Y = 0$ if the system is regular, we get at once

Theorem 3.3:

If prices of two regular single product systems coincide at $n+1$ levels of the rate of profit, the two systems are identical.

There are examples of irregular systems which are different and yet yield the same prices at all rates of profit. One obtains several well known results as corollaries of theorem 3.2, e. g. that relative prices in two sector systems are monotonic functions of the rate of profit, or more importantly that two (n, n) single product systems cannot have more than n switchpoints in common, if they are different (a switchpoint is a level of the rate of profit where all prices of two systems — and not only just the real wage — coincide).

This would suggest that two systems must be the more similar the more switchpoints they have in common, or, to put it the other way round, it would seem quite logical from a mathematical point of view to suppose that two systems which are really different cannot have two switchpoints in common, except by a fluke. However, we shall prove that reswitching is not a *mathematical* exception.

4. Reswitching and the Technology Set

The term reswitching has not always been used in the same sense. We shall solely consider the case where only the method of production for one of the commodities in the system is subject to change, i. e. where e. g. the techniques for the production of commodities $2, \ldots, n$ are given and fixed while alternative methods are available for the technique used in the production of commodity 1. Reswitching then means the possibility that a technique used in the production of commodity 1 may be eligible at two different levels of the rate of profit, separated by ranges of the rate of profit where different techniques are eligible.

If only one alternative technique exists, this may be formalized as follows: Let a productive, indecomposable single product system with input matrix A and labour vector l be given. The method of production for commodity i is (a_i, l_i), $i = 1, \ldots, n$, where a_i is the vector of physical inputs and l_i the labour input to process i. Reswitching will take place if there are two rates of profit r_1, r_2 where a second technique for the production of commodity 1 (denoted by input vector a_0 and labour input l_0) is as profitable as the original technique (a_1, l_1). I. e. the equation $(1+r)\,a_0\hat{p}+l_0=(1+r)\,a_1\hat{p}+l_1$ must hold at two rates of profit r_1, r_2.

It is useful to begin the discussion with this narrowest possible definition of reswitching.

The condition for reswitching can be rewritten as

$$(a_1-a_0)\,(1+r)\,\hat{p}\,(r)+(l_1-l_0)=0.$$

Reswitching will therefore take place if a technique (a_0, l_0) exists such that $c=(a_1, l_1)-(a_0, l_0)$ is orthogonal to the $(n+1)$-column vector

$$\tilde{p}\,(r)=\begin{bmatrix}(1+r)\,\hat{p}\,(r)\\ 1\end{bmatrix}$$

for two different rates of profit with

$$(a_0, l_0)=(a_1, l_1)-c\geqq 0.$$

Whether reswitching takes place will thus depend on the availability of an alternative technique for the production of commodity 1 on the one hand and on the shape of the curve $\tilde{p}\,(r)$ on the other. We discuss these two in turn beginning with a theorem about $\tilde{p}\,(r)$.

Theorem 4.1:

$\tilde{p}\,(r)$ takes on $n+1$ linearly independent values in $(n+1)$-dimensional space at $n+1$ different rates of profit r_i (where $r_i \neq R_j$ for all i, j), if and only if the system is regular.

Proof: Using the notation of theorem 1.1 one defines the $(n+1)$-columnvector

$$\bar{u}\,(r)=\det\,[B-(1+r)\,A]\begin{bmatrix}(1+r)\,T\,\hat{p}\,(r)\\ 1\end{bmatrix}$$

and the matrix

$$\bar{U}=[\bar{u}\,(r_1),\ldots,\bar{u}\,(r_{n+1})].$$

As in the proof of theorem 1.1 one shows firstly that $\bar{U}$ is singular if and only if there is a vector $x\neq 0$ such that $x\,\bar{u}\,(r)\equiv 0$ and secondly that the first $n-1$ components of the vector x must be zero. It is then clear that the last component of x vanishes as well. The rest is analogous to the proof of theorem 1.1.

q. e. d.

This theorem is closely related to the following: If a system is not regular, the price vector moves always within one fixed $(n-1)$-dimensional hyperplane containing the origin. But if the system is regular we have:

Theorem 4.2:

The price vectors $\hat{p}(r_i)$ belonging to $n+1$ different rates of profit $r_1, \ldots, r_{n+1}$, $r_i \neq R_j$ for all i and j, are never on the same $(n-1)$-dimensional hyperplane in n-dimensional space.

Proof: The $n+1$ points $\hat{p}(r_i)$, $i=1, \ldots, n+1$ are on a $n-1$-dimensional hyperplane in n-dimensional space if and only if there is a vector $(\lambda_1, \ldots, \lambda_{n+1}) \neq 0$ such that $\sum_{i=1}^{n+1} \lambda_i \hat{p}(r_i) = 0$ with $\sum_{i=1}^{n+1} \lambda_i = 0$.

This will be the case if and only if the matrix

$$\bar{U} = \begin{bmatrix} \hat{p}(r_1), \ldots, \hat{p}(r_{n+1}) \\ 1 \quad , \ldots, \quad 1 \end{bmatrix}$$

is singular. But $\bar{U}$ is not singular for regular systems, the proof being analogous to that of the preceding theorem.

q. e. d.

These two theorems emphasize again the erratic character of the movement of prices in function of the rate of profit in regular systems. Two values of $\tilde{p}(r)$ will never be proportionate at two different levels of the rate of profit, $n+1$ values of $\tilde{p}(r)$ will never be in the same n-dimensional hyperplane containing the origin in regular systems; n values of $\hat{p}(r)$ will be linearly independent and the $(n-1)$-dimensional hyperplane spanned by them will never contain any $(n+1)$st value of $\hat{p}(r)$ in regular systems (except when r is equal to an eigenvalue).

We have noted above that reswitching occurs if and only if there is a $(n+1)$-vector c such that $c\tilde{p}(r) = 0$ at two levels of the rate of profit, where c has to correspond to a feasible technique, i. e. $(a_0, l_0) = (a_1, l_1) - c \geqq 0$. We can now see that irregular systems are characterized by the existence of a vector c such that $c\tilde{p}(r) \equiv 0$. If c corresponds to a feasible technique, two techniques are compatible at all rates of profit. This is not really reswitching, but rather an indication of the odd and exceptional character of irregular systems: all points on the wage curve are "switchpoints" for (a_0, l_0) and (a_1, l_1). If we have the extreme case of an irregular system, i. e. if *prices equal values*, we find that *an alternative technique is compatible with the original technique either at all* *rates of profit or at most at one.* This can be seen from the fact that an alternative technique (a_0, l_0) will fulfill the equation

$$(1+r)\, a_0\, u + (1-r/R)\, l_0 = u_1$$

(where u is the vector of values and $(1-r/R)=w$ the wage rate) either identically or only for one rate of profit.

Reswitching in the sense that two techniques are equally profitable at two and only two rates of profit is therefore ruled out if prices are equal to values. We shall now show that the possibility of reswitching is characteristic for regular systems. However, whether it really takes place depends also on the alternative techniques which are available. Suppose that more than just two methods for the production of commodity one exist. If the most drastic neoclassical assumptions about technology are made, no reswitching can take place. For if we assume

1) constant returns to scale,

2) a technique for the production of one unit of commodity one is represented by a point in the non-negative orthant in R^{n+1} (where the first n components denote the amounts of required raw materials and the last the required amount of labour; the labour coefficient is always positive),

3) the feasibility of (a_0, l_0) implies the feasibility of all (a, l) where $(a, l) \geqq (a_0, l_0)$, and if we denote this $(n+1)$-dimensional technology set by TS and assume

4) strict convexity,

5) smoothness of the boundary of TS (the "technology frontier" BTS);

it is clear that a technique $(a_0, l_0) \in TS$ is eligible at a given rate of profit r, if $(a_0, l_0)\, \tilde{p}\,(r)$ is a minimum over all $(\bar{a}_0, \bar{l}_0) \in TS$. Eligible techniques are on the technology frontier BTS. The existence of a switchpoint (which is not an inner point of TS and therefore irrelevant) implies that $\tilde{p}\,(r)$ is orthogonal to some point c_0 on the (smooth) boundary $BT'S'$ of the set $\{x \in R^{n+1} | x = (a_1, l_1) - y, y \in TS\}$. Now the smoothness of $BT'S'$ insures that the normal is well defined in any point on the surface $BT'S'$ and since $\tilde{p}\,(r)$ never assumes twice the same value, *re*switching is impossible. Strict convexity insures, moreover, that there can be never more than one eligible technique corresponding to a given level of the rate of profit.

This reasoning looks persuasive and is effective in ruling out reswitching. However, it misses a very important point. A technique

(a_0, l_0) in the technology set TS which is eligible at any rate of profit is either on the boundary of the non-negative orthant $R_+{}^{n+1}$ or it is an inner point. But the latter is very unlikely, because no technique for the production of one commodity in an economy has ever been seen which used positive quantities of *all* basic commodities known in that economy as inputs. If strict convexity seems to suggest that an inner point of $R_+{}^{n+1}$ could ever be eligible as the most profitable technique for a given rate of profit, strict convexity is a dubious assumption. It is safe to assume that there will always be some zeros in the rows denoting the inputs of raw materials.

One might try to defend strict convexity by arguing somehow that a convex combination of the inputs to two different techniques for the production of the same commodity is technically superior to either of the techniques.

This argument may be justifiable for the combinations of some processes when the inputs to be combined are tools. (It is harder to find good examples when the inputs to be combined are raw materials, since the use value of a commodity frequently changes when the raw materials from which it is made are replaced by substitutes.) It is in fact possible to produce planks by means of a saw and by means of a hatchet, and perhaps advantageously with a combination of both.

But it is usually overlooked that strict convexity requires much more in the context of conventional neoclassical assumptions such as (1)—(3) above. For assumption (3) (free disposal) implies that a ton of steel — if it can be produced by means of a ton of coal and twenty man-hours — can also be produced by means of a ton of coal, twenty man-hours, 500 cherries and six elephants, since the latter two inputs may be "disposed of". Now there is no reason to assume that strict convexity obtains for a combination of these two "techniques" (in the same way as it obtained for a combination of saw and hatchet), for this would imply that a process of production could be made more productive by adding just any arbitrary input: the same amount of steel could be produced, using less coal and a few cherries more. Strict convexity implies, together with (3), that every input is a substitute for every other. Assumptions (3) and (4) are therefore not compatible in general. The assumption of a strictly convex $(n+1)$-dimensional technology set looks relatively innocent, since convexity is plausible, and so is the assumption of free disposal. Strict convexity then looks like an analytically useful additional hypothesis. But one should bear in mind that since no real process of production uses positive quantities of all commodities as inputs, the technology set is $(n+1)$-dimensional only because of assumption (3). For even if we admit assumptions (1) and (2) and suppose that TS is convex, we can hardly expect to be able to construct a feasible technique with all input coefficients positive, since we cannot expect to find for all i a technique $(\bar{a}_0, \bar{l}_0)$ which has a positive i-th component to every technique whose i-th input coefficient is zero. Hence we must conclude that TS would probably contain no positive point at all if we did not have assumption (3). The economically relevant techniques which do not contain disposable inputs are therefore all contained in the boundary of the non negative orthant $R_+{}^{n+1}$. They form a set which is less than $(n+1)$-dimensional. This point is never properly recognized despite the prevalence of zeroes in all empirical Leontief systems, because economists are used to think in terms of two or three sector models where a positive vector of inputs looks innocent.

It is nevertheless analytically convenient to retain free disposal (assumption (3)). There is no harm either in assuming the possibility of perfect substitutability for some groups of inputs, but general substitutability must not be assumed. Assumption (4) (strict convexity) is therefore to be replaced by the assumption of convexity. (The difference is fundamental, as we shall see.) Instead of (5), we introduce a new assumption: a technique $(a_0, l_0) \in TS$ will be said to contain no disposable inputs if none of the input coefficients of (a_0, l_0) can be reduced without increasing another. We assume that every technique without disposable inputs contains at least one coefficient which is zero. Our reasoning then implies that every point on the boundary of the technology TS, the technology frontier TS, should be assumed to be spanned by a set of points (a_0, l_0) where at least one of the components of each vector a_0 vanishes, or TS is obtained by adding disposable inputs to such points. The boundary of TS will therefore not be smooth where it intersects the boundary of $R_+{}^{n+1}$. Strict convexity may obtain on the boundary of $R_+{}^{n+1}$ when two or even several inputs are substituted for each other, but cannot be expected to obtain with *all* substitutions and not for inner points of $R_+{}^{n+1}$.

Now it is important to note that an inner point of $R_+{}^{n+1}$ can at best be eligible at any given rate of profit by a fluke. For $(a_0, l_0)\, \tilde{p}\, (r)$ can be a minimum for given r only if either (a_0, l_0) is on the boundary of $R_+{}^{n+1}$. Since no uniquely defined tangency plane of BTS exists in the boundary point, (a_0, l_0) will then in general remain eligible, except in fluke cases, if a small variation of the rate of profit takes place. Or (a_0, l_0) is an inner point of $R_+{}^{n+1}$. We abstract first from the existence of disposable inputs. In this case, (a_0, l_0) is a convex combination of at least two points (a_0', l_0') (a_0'', l_0'') which are on *BTS* and on the boundary of R_+^{n+1}, so that $(a_0, l_0)\, \tilde{p}\,(r)$ can be a minimum for given r only if $(a_0, l_0)\, \tilde{p}\,(r) = (a_0', l_0')\, \tilde{p}\,(r) = (a_0'', l_0'')\, \tilde{p}\,(r)$. Now any arbitrarily small variation of r in a *regular* system implies (because of theorem 4.1 above) that either $(a_0', l_0')\, \tilde{p}\,(r)$ or $(a_0'', l_0'')\, \tilde{p}\,(r)$ will become smaller than $(a_0, l_0)\, \tilde{p}\,(r)$. Hence (a_0, l_0) can be eligible at r only if (a_0, l_0) and (a_0'', l_0'') are switchpoints at r, hence only by a fluke. This explains why even if two techniques with some zero coefficients exist such that their linear combinations are positive, their joint use will not be observed.

But in an economy involving many commodities and processes it is likely (though we do not assume it) that not even groups of processes involving no disposable inputs will exist such that their convex combinations are positive. The boundary of the technology set *BTS* contains then positive points only because of the free disposal assumption. The positive points of *BTS* will therefore consist of pieces of hyperplanes which are parallel to at least one of the coordinate axes, and it follows that none of these points will ever be eligible in this case since $\tilde{p}\,(r)$ is positive.

These considerations may seem to imply that linear activity analysis provides a better representation of technology than the above set theoretical description, since linear activity analysis is based on the assumption of a finite number of constant returns to scale techniques. However, I do not want to exclude the possibility of continuous substitution altogether. Continuous or even differentiable substitution possibilities may obtain with pairs of groups of inputs. But if the technology frontier is not strictly convex everywhere, techniques on its boundary *BTS* will become eligible in discontinuous succession.

Thus we find that, as r varies in a regular system between zero and R, different techniques of *BTS* will become eligible. They will be on the boundary of R_+^{n+1}, or, if they are not, they are spanned by techniques which are eligible at the same rate of profit and which are on the boundary of R_+^{n+1}. In our discussion of the possibility of reswitching we may thus assume that the relevant eligible techniques are not inner points of R_+^{n+1}, and therefore that the technology frontier is not smooth in the neighbourhood of the relevant eligible points which are on the boundary of R_+^{n+1}.

As soon as edges and corners are admitted in the technology set, reswitching can easily occur in regular systems even if convexity is retained, for although the price vector will never assume the same value at two different levels of the rate of profit, its erratic behaviour may easily make it possible that the same corner will be profitable at two different levels of the rate of profit while another may be profitable in between.

Our results about the behaviour of the price vector and our discussion of the technology set will now allow us to give more precision to the statement that reswitching is "easily possible".

To begin with, we assume again that only one technique (a_0, l_0) for the production of one unit of commodity one exists which is an alternative to the actual technique (a_1, l_1). Suppose that the two techniques are different and equally profitable at $r=r_1$. How likely is it that a rate of profit $r_2 \neq r_1$, $0 \leqq r_2 < R$, can be found such that both techniques are equally profitable at $r=r_2$? We assume that the system is regular, for if prices are equal to values, reswitching is ruled out, and intermediate cases of irregular systems present uninteresting complications.

It is, of course, not possible to give an exact measure for the likelihood of reswitching in this case. But we can at least argue why reswitching is not just a mathematical fluke by considering the set

$$Y(r_1) = \{(a, l) \geqq 0 \mid (a, l)\, \tilde{p}(r_1) = (a_1, l_1)\, \tilde{p}(r_1)\}$$

of all "potential" techniques or vectors which are formally equally profitable as technique (a_1, l_1) at $r=r_1$ and the subset of $Y(r_1)$

$$Z(r_1) = \{(a, l) \in Y(r_1) \mid (a, l)\, \tilde{p}(r_2) = (a_1, l_1)\, \tilde{p}(r_2) \text{ for some } r_2 \neq r_1,\ 0 \leqq r_2 < R\}$$

which consists of all "potential" techniques or vectors which are as profitable as (a_1, l_1) at the given rate of profit r_1 and at *some* other rate of profit $r_2 \neq r_1$. That is to say, $Z(r_1)$ is the set of potential techniques which have one switchpoint with (a_1, l_1) at r_1, and another at some r_2 where r_2 is not the same for all points of $Z(r_1)$. Obviously, reswitching is a mathematical fluke if $Z(r_1)$ is only a "very small part" of $Y(r_1)$, for if the set of "potential" reswitchpoints $Z(r_1)$ is small in relation to the set of "potential" switchpoints $Y(r_1)$, the only actual alternative technique (a_0, l_0) — which is in $Y(r_1)$ — will only by a fluke be to be found in the set $Z(r_1)$. While I have not been able to construct an exact measure of $Z(r_1)$ in relation to some economic property of the system, one can at least prove:

Theorem 4.3:

The n-dimensional measure of $Z(r_1)$ as a percentage of the n-dimensional measure $Y(r_1)$ is positive, if the system is regular, and zero, if prices are equal to values.

Proof: If prices are equal to values u, prices are proportional to the vector of direct labour inputs (section 2 above), and "reswitching" of any potential technique (a_0, l_0) implies that (a_0, l_0) is compatible with (a_1, l_1) at all rates of profit (this section 4 above). Therefore $l_0 = l_1$, $a_0 u = a_1 u$ and $Z(r_1)$ is an $(n-1)$-dimensional set.

If the system is regular, the set

$$Y(r) = \{(a, l) \geqq 0 \mid [(a, l) - (a_1, l_1)] \tilde{p}(r) = 0\}$$

is an n-dimensional simplex in R_+^{n+1} spanned by its $n+1$ corner points on the coordinate axes of R_+^{n+1}. We have $(a_1, l_1) \in Y(r)$ for all r. Clearly

$$Z(r_1) = \bigcup_{\substack{0 \leqq r < R \\ r \neq r_1}} \{Y(r) \cap Y(r_1)\}$$

Because of Theorem 4.1, the simplices $Y(r)$ and $Y(r_1)$ have no corners in common except for a finite number of rates of profit r, $0 \leqq r < R$, $r \neq r_1$. Yet $Y(r) \cap Y(r_1)$ is not empty since $(a_1, l_1) \in \in \{Y(r) \cap Y(r_1)\}$. (a_1, l_1) cannot be expected to be a positive vector. but $Y(r)$ and $Y(r_1)$ must have positive points in common.

To see this, denote the corners of $Y(r_1)$ by $\xi_i e_i$, $i = 1, \ldots, n+1$, (where e_i is the i-th unit vector) and the corners of $Y(r)$ by $\eta_i e_i$, $i = 1, \ldots, n+1$. (a_1, l_1) cannot be a corner of either $Y(r)$ or $Y(r_1)$ since the system is regular. Since $(a_1, l_1) \in \{Y(r) \cap Y(r_1)\}$ we can neither have $\xi_i < \eta_i$, $i = 1, \ldots, n+1$, nor $\xi_i > \eta_i$, $i = 1, \ldots, n+1$. Without loss of generality $\xi_i < \eta_i$, $i = 1, \ldots, t$, and $\xi_i > \eta_i$, $i = t+1, \ldots, n+1$; $1 \leqq t \leqq n$. The straight lines $\lambda \xi_i e_i + (1-\lambda) \xi_j e_j$ and $\lambda \eta_i e_i + + (1-\lambda) \eta_j e_j$, $0 \leqq \lambda \leqq 1$, have one point $h_{i,j}$ in common for all pairs i, j with $1 \leqq i \leqq t$, $t+1 \leqq j \leqq n+1$. The $t(n+1-t)$ $(n+1)$-vectors h_{ij} and their convex combinations are in $Y(r) \cap Y(r_1)$, hence any convex combination with positive coefficients yields a positive point in $Y(r) \cap Y(r_1)$.

$Y(r) \cap Y(r_1)$ is therefore a $(n-1)$-dimensional set containing a positive point in R_+^{n+1}. We have to show that the correspondence $r \to \{Y(r) \cap Y(r_1)\}$, $0 \leqq r < R$, $r \neq r_1$, covers a n-dimensional subset of $Y(r_1)$ containing an open n-dimensional set.

To prove this we note that the points on $Y(r_1)$ covered by the correspondence $r \to \{Y(r) \cap Y(r_1)\}$ are for sufficiently small variations of r points which are also covered by the mapping

$$\phi: (r, \varrho_2, \ldots, \varrho_n) \to R_+^{n+1}$$

given by

$$(a, l) = \{(a_1, l_1) \tilde{p}(r), (a_1, l_1) \tilde{p}(r_1), \varrho_2, \ldots, \varrho_n\} \{M(r)\}^{-1}$$

where

$$M(r) = \{\tilde{p}(r), \tilde{p}(r_1), \tilde{p}(r_2), \ldots, \tilde{p}(r_n)\}$$

with $r, r_1, \ldots, r_n$ all different, $0 \leqq r_i < R$, and where $\varrho_2, \ldots, \varrho_n$ are parameters, varying between zero and $+\infty$. Conversely: if $(a\ l) = \phi(r, \varrho_2, \ldots, \varrho_n)$ and if $(a, l) \geqq 0$, we have $(a, l) \in \{Y(r) \cap Y(r_1)\}$. There is a point $(\bar{a}, \bar{l}) \in Y(r_1) \cap Y(r)$, $r \neq r_1, \ldots, r_n$, such that $(\bar{a}, \bar{l}) > 0$. With $(\bar{a}, \bar{l})\ \tilde{p}(r_i) = \bar{\varrho}_i > 0$, $i = 2, \ldots, n$, the mapping ϕ maps the point $(r, \bar{\varrho}_2, \ldots, \bar{\varrho}_n) \in R^n$ onto the positive point $(\bar{a}, \bar{l})$ on $Y(r_1)$. Since the image point of ϕ is positive and since ϕ is continuous and one-to-one[9] in a sufficiently small n-dimensional neighbourhood of $(r, \bar{\varrho}_2, \ldots, \bar{\varrho}_n)$, the correspondence $r \to Y(r) \cap Y(r_1)$ covers an open n-dimensional set in $Y(r_1)$ for small variations of r.

q. e. d.

The geometry of $Z(r_1)$ increases in complexity with n, i. e. with the number of commodities. If $n = 2$, $Y(r)$ is a two dimensional simplex in R_+^3 and $Z(r)$ can be drawn. The triangle in Fig. 1 represents $Y(r_1)$, the shaded area $Z(r_1)$.

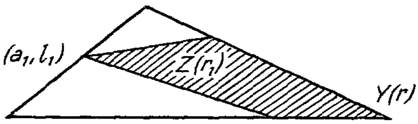

Fig. 1. The triangle represents the set of potential techniques which are as profitable as the actual technique (a_1, l_1) at rate of profit r_1 and $Z(r_1)$ represents the set of potential techniques which are as profitable as (a_1, l_1) also at some other rate of profit

$Z(r_1)$ degenerates to a straight line if prices are equal to values. The area of $Z(r_1)$ is the greater the more directions in space are

[9] There are in fact exceptional points. If the mapping is not one to one in $(\bar{a}, \bar{l})$, it means that ϕ maps some point $(r', \varrho_2', \ldots, \varrho_n') \neq (r, \varrho_2, \ldots, \varrho_n)$ onto $(\bar{a}, \bar{l})$. It follows at once that $\bar{\varrho}_i = \varrho_i'$, $i = 2, \ldots, n$, since only the first column of matrix $M(r)$ varies with r. Therefore, since $(\bar{a}, \bar{l})\ \tilde{p}(r) = (a, l_1)\ \tilde{p}(r)$, $(a, \bar{\bar{l}})\ \tilde{p}(r') = (a_1, l_1)\ \tilde{p}(r')$, and $(\bar{a}, \bar{l})$ turns out to be a switchpoint with (a_1, l_1) not noly at r and r_1, but also at r'. One has therefore to choose $(\bar{a}, \bar{l})$ such that it does not happen to be in the $(n-2)$ dimensional subset $Y(r) \cap Y(r_1) \cap Y(r')$ of $Y(r) \cap Y(r_1)$ for any r' in a sufficiently small neighbourhood of r. This will always be possible since $Y(r) \cap Y(r_1) \cap Y(r')$ does not cover more than a small part of $Y(r) \cap Y(r_1)$ if r' varies only a little. All these arguments depend crucially on the regularity of the system.

assumed by the vector $\tilde{p}(r)$, $0 \leqq r < R$. $Z(r_1)$ can never cover the whole of $Y(r_1)$, however. No point on the $(n+1)$ $-st$ coordinate axis belongs to $Z(r)$ since the equation $(0, \bar{l})\ \tilde{p}(r) = (a_1, l_1)\ \tilde{p}(r)$, i. e. the equation $\bar{l} = \hat{p}_1(r)$ is fulfilled for at most one rate of profit.

The larger the area $Z(r_1)$ of potential techniques which lead to reswitching, the greater the likelihood that the only actual alternative technique $(a_0, l_0) \in Y(r_1)$ will be in $Z(r_1)$.

So far, we have assumed that only one actual alternative technique (a_0, l_0) was available. More precisely, the technology frontier was spanned by (a_1, l_1) and (a_0, l_0). r_1 was the rate of profit at which (a_0, l_0) was as profitable as the original technique (a_1, l_1). In order to determine wether (a_0, l_0) was likely to lead to reswitching we looked at the set $Z(r_1)$ of all potential techniques which are alternatives to (a_1, l_1) at $r = r_1$ and at some other rate of profit. Since the set $Z(r_1)$ was found to be of the same dimension as the set $Y(r_1)$, the likelihood of (a_0, l_0) being in $Z(r_1)$ was not negligible. No theorem could be proposed expressing the measure of $Z(r_1)$ as a percentage of the measure of $Y(r_1)$ in function of some economic property of the system; all that could be said for sure was that the measure of $Z(r_1)$ as a percentage of the measure of $Y(r_1)$, hence the likelihood of (a_0, l_0) being in $Z(r_1)$, was not zero.

This remains true, if the technology frontier is spanned by more than two points, although we get a complication.

An alternative technique (a_0, l_0) which is as profitable as (a_1, l_1) at r_1 with both techniques being eligible at r_1, may not be eligible any more at some other rate of profit $r_2 \neq r_1$ at which (a_0, l_0) is as profitable as (a_1, l_1). There will then be reswitching in that the two techniques are equally profitable at r_1 and at r_2, but they are eligible only at r_1. Both are less profitable at r_2 than some third technique. By making sufficiently bold assumptions about the technology frontier one can ensure that whenever two techniques are equally profitable at two different rates of profit, there will always be a third technique dominating them by being more profitable than either at one of the two rates of profit. In the extreme case, if strict convexity and smoothness of the frontier are assumed, one gets rid of reswitching in so far as there is then only one eligible technique at each rate of profit.

But I have tried to show that strict convexity and smoothness are highly dubious assumptions. If corners of the technology frontier are admitted, it may still be that whenever two corners are equally profitable at two rates of profit, a third will be eligible at one of these two rates of profit. But there is no reason why this should be so in general. If (a_0, l_0) is as profitable as (a_1, l_1) at $r=r_1$, both being eligible, there is a positive possibility that (a_0, l_0) will be in $Z(r_1)$, i. e. as profitable as (a_1, l_1) at some $r=r_2$. And if (a_0, l_0) is in $Z(r_1)$, there is again surely a positive possibility that (a_0, l_0) is not dominated by a third technique at $r=r_2$. The fact that the product of two probabilities may be a probability smaller than either does not reduce a possibility to a fluke.

5. Wicksell Effects

We may try to pursue the pure logic of a Sraffa system a little further by applying the formula derived above for systems with simple roots to the analysis of Wicksell effects. This will be all the more useful since perverse movements of the capital labour ratio are at least as relevant for the criticism of neoclassical theory as reswitching, and in discussing them we have the advantage of not having to make any assumption about alternative techniques, be it in the conventional form of Neoclassical assumptions about a strictly convex technology set or, more cautiously but still hypothetically, a book of blue prints[10].

We calculate the capital labour ratio of a stationary non basic system where the non-basics are all pure consumption goods, *viz.* they are produced by means of labour and basics alone. The basic part of the system is assumed to be regular with no multiple roots in the characteristic equation. The system is supposed to produce a surplus of non-basics only, and the basket of non-basics in the surplus will be taken as the numéraire for prices. The model represents the obvious generalisation of the conventional two-sector system with one basic commodity and one non-basic serving as "numéraire". Formally this may be expressed as follows:

The input matrix is given by

$$A=\begin{bmatrix} A_1{}^1 & 0 \\ A_2{}^1 & 0 \end{bmatrix}$$

where $A_1{}^1$ is a (n, n) indecomposable matrix for the basic part of

[10] The book of blue prints may always contain goods which may become commodities in the new system if the new technique is adopted while they were not commodities in the old system. This is an awkward possibility since it implies that we must be able to list and measure goods which have not been listed or measured or even been defined as separate goods by the market. There is no such methodological difficulty involved when we calculate what prices of production of *given* commodities would be if the rate of profit changed with techniques remaining unchanged.

the system and A_2^1 a (m, n) matrix. The output matrix (unit matrix) I and the labour vector l are partitioned accordingly. The net surplus of consumption goods to be produced is given by a $(n+m)$-row vector $d=(d_1, d_2)$ where the n-vector d_1 equals zero and where the m-vector $d_2>0$. Activity levels q are then given by

$$q = d\,(I-A)^{-1} = [d_2 A_2^1\,(I_1^1 - A_1^1)^{-1}, d_2].$$

Total labour in the economy is taken to be unity, i. e. $ql=1$. The price equations are

$$p_1 = (1+r)\,A_1^1 p_1 + w l_1$$
$$p_2 = (1+r)\,A_2^1 p_1 + w l_2$$

where $dp \equiv 1$ so that

$$w\,(r) = \frac{1}{d_2 \hat{p}_2\,(r)}\,;\ \hat{p} = \frac{p}{w}.$$

The capital labour ratio is

$$k = \frac{K}{L} = \frac{q\,Ap}{w\,q\,l} = \frac{q\,A^1 \hat{p}_1}{d_2 \hat{p}_2}.$$

The n eigenvectors of A_1^1 are linearly independent so that the sum of the inputs of basics in the processes of non basics can be represented as a linear combination of them

$$d_2 A_2^1 = \sum_{i=1}^{n} \lambda_i q_i$$

where

$$(1+R_i)\,q_i A_1^1 = q_i,$$

and where we normalize the q_i to $q_i l_1 = \frac{1}{1+R_i}$ (since the system is regular, we have $q_i l_1 \neq 0$).

Using

$$q_i\,[I - (1+r)\,A] = q_i \left(1 - \frac{1+r}{1+R_i}\right)$$

the formula for the capital labour ratio may be simplified

$$k = \frac{d_2\,A_2^1\,(I_1^1 - A_1^1)^{-1}\,A_1^1\,\hat{p}_1 + d_2\,A_2^1\,\hat{p}_1}{d_2\,(1+r)\,A_2^1\,\hat{p}_1 + d_2\,l_2}$$

$$= \frac{\sum\limits_{i=1}^{n} \lambda_i \left(\frac{1+R_i}{R_i}\,\frac{1}{1+R_i}\,\frac{1+R_i}{R_i - r} + \frac{1+R_i}{R_i - r}\right) q_i l_1}{d_2 l_2 + (1+r)\,\sum\limits_{i=1}^{n} \lambda_i \frac{1+R_i}{R_i - r}\,q_i l}$$

so that it is shown to depend essentially on the "eigenvalues" R_i and the coefficients λ_i by means of which the inputs of basics to the processes of non basics are expressed as a linear combination of the eigenvectors:

$$k = \frac{\sum_{i=1}^{n} \lambda_i \frac{1+R_i}{R_i} \frac{1}{R_i - r}}{d_2 l_2 + \sum_{i=1}^{n} \lambda_i \frac{1+r}{R_i - r}}.$$

The capital labour ratio is thus represented as a rational function of r in explicit form. It reduces to a constant in essentially only one case: if by coincidence $d_2 A_2^1 = \lambda_1 q_1$ and, also by coincidence, $d_2 l_2 = \lambda_1$ we get $k = 1/R_1$. The same simple result is obtained in a basic Sraffa system (where the surplus consists of basics only) in standard proportions and also in a basic Sraffa system where prices are equal to values. Here, where non-basics are involved, the situation is more complicated, but one can easily show that $d_2 l_2 = \lambda_1$ implies that the organic composition is the same in both sectors of a two sector model where the first sector produces a basic good by means of itself and labour and where the second sector produces a non-basic by means of the output of the basic sector and labour.

If the vector of inputs of basics to non-basic industries does not happen to be proportional to the standard commodity of the basic part of the system and if the coefficient of proportionality does not happen to be equal to total labour employed in non-basic industries, the capital labour ratio may vary in almost any conceivable way with the rate of profit. The point is that these variations are due to the structure of the basic part of the system, for the formula shows that the capital labour ratio of the entire system depends crucially on the eigenvalues and the eigenvectors of the basic part of the system. This result confirms the thesis the Wicksell effects are mainly due to the interaction of the basic industries. It is therefore out of place to discuss, as is often done, reswitching or Wicksell effects in terms of two sector models with one basic and one consumption good, for the relevant problems of capital theory are visible only in models involving several basic goods.

6. The Capital-Wages Ratio

The conclusion of the previous section is, when separated from the argument supporting it, not very impressive. To say that reswitching may take place or may not take place, or that the capital labour ratio may move in either direction except if it is by coincidence constant is very nearly an empty statement which can be important only as a warning to those who are still trying to get round the criticisms made against neoclassical theory by means of some clever and artificial construction. Recently Kazuo Sato[11] has claimed in the Quarterly Journal of Economics that "the neoclassical postulate ... remains one of the most powerful theorems in economic theory" (p. 355). He supports this claim by enlarging on Professor Samuelson's construction of a surrogate production function. He takes Samuelson's old two sector model[12] without assuming as Samuelson did that the organic compositions in both sectors are the same. He is nevertheless able to show that reswitching will not occur provided sufficiently bold assumptions about available techniques are made, i. e. provided the existence of a "technology frontier" is assumed and provided the substitution properties of the technology frontier are appropriate. His article is a very nice exercise in the analysis of two sector models with variable techniques, but I hope to have reminded the reader (as Sato himself is sufficiently candid to admit) that the real difficulties of capital theory begin when we are dealing with a many sector model, i. e. essentially when we are dealing with a model involving several *basic* goods. Our model provides a critique of Sato, since it is a direct generalisation of Sato's version of Samuelson's two sector model, in that basics are here the only inputs to production besides labour and in that the non-basics furnish the standard of prices.

This article has confirmed that prices of production follow a twisted curve in function of the rate of profit in regular systems involving several basic industries. The consequent complicated movement of prices is excluded only if prices are equal to values. It is "evened out" for the standard commodity. In the general case it is such that reswitching becomes an irrefutable possibility if it is recognized that the technology frontier is likely to have corners. And even if the technology frontier is assumed to be smooth, there will still be Wicksell effects for a *given* technique. It is therefore no wonder when people complain that the reswitching controversy has made capital theory awfully difficult. However, I want to conclude with a more constructive remark. The difficulties with the

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[11] Kazuo Sato; op. cit.

[12] P. A. Samuelson: Parable and Realism in Capital Theory: the Surrogate Production Function, Review of Economic Studies 39 (1962), pp. 192—206.

capital labour ratio are in part due to the fact that it is a hybrid concept in that the measurement of capital requires a measurement in terms of absolute prices while labour is measured in terms of physical units. If instead of the capital labour ratio we use the (perhaps pseudo-marxist) concept of an "organic composition of capital expressed in price terms" we get rid of the problem of choosing an appropriate standard of prices. The "organic composition of capital in price terms" is simply the relation of total capital to total wages in the economy: K/W. Given the technique and the labour force this expression depends on the rate of profit. But it does not depend on the chosen standard of prices because the price standard occurs both in the numerator and the denominator. Moreover, the capital-wages ratio is at least a monotonic function of the rate of profit in a single product system. In formulas:

$$\frac{K}{W} = \frac{qAp}{wql} = \frac{qA\hat{p}}{ql}$$

where q denotes activity levels, p, w prices in terms of any standard and $\hat{p}$ prices in terms of the wage rate. Since prices in terms of the wage rate rise monotonically with the rate of profit for a given technique, the capital wages ratio will do the same (between zero and the maximum rate of profit).

The capital-wages ratio is the relevant concept from the microeconomic point of view when the entrepreneur wishes to assess the relative *cost* of capital and labour; when he wants to compare "capital" and labour in physical terms he has to compare machines, raw materials and men. The concept of the capital-wages ratio is equally useful in macroeconomics since it relates the distribution of income between profits and wages P/W with the rate of profit P/K:

$$r = \frac{P}{K} = \frac{P/W}{K/W}.$$

If the curve indicating the capital-wages ratio in function of the rate of profit for a given technique (the "capital-wages function") shifts upwards or downwards because of a technological change (technical progress), it follows that the rate of profit is lowered or raised accordingly if the distribution of income (P/W) is fixed. This conclusion which is important for any discussion of the interdependence of income distribution, technical progress and the rate of profit may be drawn because the capital-wages function is (for a given technique) a monotonically rising function of the rate of profit, and because it is, moreover, a pure number, i. e. independent of the monetary standard of prices[13] (see Fig. 2).

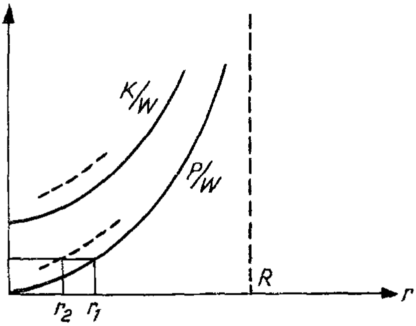

Fig. 2. The capital-wages ratio and the profits-wages ratio as functions of the rate of profit (capital-wages function and profit-wages function respectively). If the capital-wages function shifts upwards because of technological change, the profit-wages function does the same. The shifting of the curves (dotted lines) entails a fall in the rate of profit from r_1 to r_2 if the actual profit wages ratio in the economy is not affected by the technological change

I believe that J. Robinson and N. Kaldor were right in asserting that the dilemma posed by the heritage of neoclassical theory can only be overcome by shifting attention from processes of substitution to technical innovation. A discussion of the macro effects of technical progress involves an analysis of the relation between "microeconomic" switches of technique in physical terms and macroeconomic changes of "factor ratios".

If such an analysis makes it possible to express the effect of "microeconomic" changes of techniques in terms of shifts of the macroeconomic capital-wages function we should get nearer to a postneoclassical theory of the interaction between progress, distribution and profitability.

Address of author: Prof. Dr. Bertram Schefold, Fachbereich Wirtschaftswissenschaften, Johann-Wolfgang-Goethe-Universität, D-6000 Frankfurt, Federal Republic of Germany.

[13] This is discussed in detail in B. Schefold: Fixed Capital as a Joint Product and the Analysis of Accumulation with Different Forms of Technical Progress; to be published.