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MARX IN THE LIGHT OF MODERN ECONOMIC THEORY[1]

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There are two types of mathematical economists, one who applies existing mathematics to economic problems (the best example is Cournot) and the other who anticipates new mathematical problems within economics. Taking Marx as the second type of economist (Section 1), I discuss two of his problems: the fundamental Marxian theorem (Section 2) and the transformation problem (Section 3). In Section 2 I propose a generalisation of the theorem to the effect that the theorem does not need the labour theory of value and hence is independent of any criticisms of that theory. In Section 3 it is seen that the transformation problem is formally identical with the Markov chain process transforming the initial position to the ergodic position.

1. INTRODUCTION

FOR THE PAST FEW YEARS I have been interested in Marx from a rather peculiar point of view. In spite of the explosive development of highbrow mathematical economics in the post-war period, it seems to me that we are still working within more or less the same paradigm as that set in the last century. The problems with which such economists as Cournot, Marx, Walras, Bohm-Bawerk, Edgeworth, Pareto, and Wicksell were concerned are still our subjects. On very many occasions our sophisticated mathematical economic analysis has done nothing more than confirm those same conclusions which they obtained from their somewhat crude mathematical economic models. If this is so, it means that they unconsciously or consciously used mathematics in an "efficient" way. And, as a mathematical economist, I naturally wanted to learn from them how they could have been so efficient.

Since joining the London School of Economics, I have been favoured with the chance to deliver a course of lectures on Marxian economics. Having this opportunity, and taking Marx as a typical representative of the 19th century economists who were strong in economics but not so advanced in mathematics as our contemporaries, I read his *Capital* with the intention of discovering the secrets of their efficiency. However, one might query whether I am correct in taking Marx as a typical sample. It is of course true that he was an outstanding economist. But it is also true that, among the economists mentioned above, Marx is unique; in fact, some might agree with Lange's contention [2] that Marx's economics was weak in analysing the effects on some economic variables of a change in one of them and its merits lay in providing a socio-economic explanation of the economic evolution of capitalist society. There may still be many economists who do not regard Marx as a mathematical economist but believe that he was against mathematical economics.

[1] This paper was presented as the Walras Lecture at the annual Econometric Society Meeting, 28 December, 1973 in New York. It also was read as my Inaugural Lecture at the London School of Economics on 15 November, 1973.

In spite of the fact that his mathematical background was not very rich and his *Capital* does not contain advanced mathematical formulas at all, I believe we should recognise Marx as a mathematical economist. In his letter to Engels dated 11 January, 1858, Marx wrote:

> I am, when working out economic principles, so annoyingly obstructed by miscalculations that, in despair, I have again set myself the task of getting through algebra quickly. Arithmetics ever remains foreign to me. By this detour through algebra, however, I quickly train myself up again [**3**, p. 256].

Also, in his letter of 6 July, 1863 to Engels, he wrote that he had been studying differential and integral calculus and enthusiastically recommended it to Engels as he thought calculus was very useful for Engels' military studies. These statements show that Marx studied algebra and calculus in order to apply them to his economic investigations. But he never used them explicitly; at first sight, his *Capital* looks as if it were mostly non-mathematical. In reading it carefully, however, we find that many of his verbal economic discussions can be translated into rigorous mathematical language. Moreover, we find that mathematical problems, even new mathematical problems, are concealed in his economics. We may indeed say that he was an intrinsic mathematician. And such a person, by his nature, would be uninterested in fitting economics to ready-made mathematics, but would certainly be interested, as Marx really was, in extending mathematics to suit economics or in investigating more basic mathematical philosophical problems, such as the foundations of the infinitesimal calculus.[2]

It is interesting to compare Marx with Cournot in their contributions to mathematical economics and their mathematical backgounds. Cournot, who taught calculus and mechanics at Lyon and wrote books on elementary calculus, probability, and algebra, was obviously far better educated in mathematics than Marx. On the other hand, as for achievements in mathematical economics, Cournot confined himself only to applying differential and integral calculus to the analysis of the behaviour of monopolistic, oligopolistic, and competitive firms, and the stability of the market equilibrium. It was a beautiful application and greatly promoted mathematical economics, especially through Walras. At the same time, however, we must admit that Cournot set no new problem for mathematics. On the other hand, because of his lack of mathematical training, Marx could not neatly apply the then existing mathematics to economic problems. Even if he had been able to do so, as I have said above, he would not have liked to devote himself to such a job; he was too ambitious and too original. Instead, by formulating economic problems precisely, he discovered new mathematical problems within economics. These problems were subsequently rediscovered independently by mathematicians and developed into important subjects in mathematics. In fact, as will be seen in this paper, Marx met the Frobenius-Perron theorem of non-negative matrices (or more precisely, the conditions which we now call the Hawkins-Simon

[2] See [**4**]. From this book, we can see that Marx was interested in such fundamental mathematical problems as differentiability, etc.

conditions) in the so-called Fundamental Marxian Theorem and the problem of the Markov chain in his Transformation Problem. Marx was often criticised because he could not correctly solve these problems. However, I believe we should not blame him at all for this failure; on the contrary, it is to his great credit that he discovered these problems before the mathematicians and obtained his own solutions. The truth of these solutions can be shown after some revisions are made by using the appropriate mathematical theorems found later.

In this respect, Marx was similar to Walras, who was also definitely less able than Cournot in mathematics, although he had a richer background than Marx. Walras, too, provides a remarkable example of a man for whom economics came earlier than mathematics. Almost at the same time as Marx was diligently tackling his problems, Walras was struggling against a similar difficulty. He was confronted with an economic problem, the existence of a competitive equilibrium, which could not be solved by the mathematics of that time, for Brouwer, the first mathematician who could solve the necessary theorem, was born only in 1881. It is interesting to see how Walras and Marx reacted to their perplexities. Walras developed the "social scientific" device of "tatonnement," which was a crude mapping of the price simplex into itself, and obtained practically the same solution as we now have by the rigorous application of Brouwer's fixed-point theorem.[3] Similarly, Marx took a social scientific approach to cover his mathematical deficiency. He invoked the classical theory of value and almost correctly solved the mathematical problems by appealing to his intuition as a social scientist.

In any case it is indeed paradoxical and instructive to see that the man with the best knowledge of mathematics did not contribute to the development of mathematics through his mathematical economic investigations, whereas the other two, who were relatively poor in mathematics, found problems anticipating new mathematical theorems, the fixed-point theorem in the case of Walras and the problem of Markov chains, etc., in the case of Marx. It is a great pity that neither Walras nor Marx had a mathematical collaborator who could solve and develop their problems in a mathematically proper and rigorous way; their mathematical economic problems had no effect on mathematics until at last Wald took up Walras' problem.

I have learned many useful, positive, and negative lessons from reading Marx's *Capital*, of which I report on the two most remarkable in this paper. However, the greatest harvest from my study was that I became convinced of Keynes' view:

> ... the master economist must possess a rare combination of gifts. He must reach a high standard in several different directions and must combine talents not often found together. He must be mathematician, historian, statesman, philosopher—in some degree. He must understand symbols and speak in terms of the general, and touch abstract and concrete in the same flight of thought. He must study the present in the light of the past for the purpose of the future [**1**, pp. 173–174].

[3] In fact, Uzawa [**11**] has shown that Walras' tatonnement, if it is rigorously reformulated, is equivalent to Brouwer's theorem; that is, we can prove one by the other and vice versa.

2. THE GENERALISED FUNDAMENTAL MARXIAN THEOREM

The central theme of Marx's *Capital* is the viability and expandability of the capitalist society. Why can and does the capitalist regime reproduce and expand itself? Obviously an immediate answer to this question would be: "Because the system is profitable and productive." Then we may ask: "Why is the system profitable and productive?" Marx gave a peculiar answer to this question, that is: "Because capitalists exploit workers."

Some of us may be unhappy with this answer, while others are enthusiastic about it. But even though one may like or dislike it ethically, I dare say it is a very advanced answer. I am not referring to its political progressiveness but its mathematical modernness. It is closely related to what we now call the Hawkins-Simon condition. It gives the necessary and sufficient condition so that the warranted rate of profit and the capacity rate of growth (defined later) are positive. Von Neumann and others examined these concepts but were not concerned with their positiveness; von Neumann was satisfied with the weaker finding that the warranted rate of profit and the capacity rate of growth are at least as large as -1.[4] However, we must verify their positiveness in order to affirm the productiveness of the capitalist system.

To tackle this truly modern problem, Marx had to go it alone. He could not ask for assistance from such mathematicians as Frobenius, Perron, or Markov, or such economists as Hawkins, Simon, or Georgescu-Roegen, simply because they had either not been born or had not yet discovered their theorems concerning non-negative matrices which were later found to be so useful in solving the problem.[5] Confronted with this very revolutionary occasion, Marx, like Walras, decided to take a "social scientific approach" instead of trying to find the new necessary mathematical theorems by himself. And, although Marx did not discover a completely new device such as Walras' tatonnement, he was a magician and put old wine into a new bottle. He was very successful in using this social scientific approach to cover his mathematical deficiency and, like Walras, obtained practically the same solution as we accept today by the rigorous application of the Frobenius-Perron theorem.

Marx used the classical labour theory of value, not as a primitive or approximately valid theory of competitive equilibrium prices as it had been used, but to calculate, in a purely technocratic way, the value or the labour-time directly or indirectly necessary to produce a unit of each commodity. He then calculated the value of the labouring power, that is, the quantity of labour necessary to reproduce itself, or equivalently, the value of the subsistence-consumption bundle of commodities, that is, the total amount of labour contained in the mass of necessaries required to produce, develop, maintain and perpetuate the labouring power. Let this value be denoted by T^*. Under the assumption that each worker is paid wages only at the subsistence level (this is Marx's basic assumption), the worker receives the mass of necessaries containing T^* hours of labour by working for T hours a day.

[4] When the economy is "indecomposable," the warranted rate of profit equals the capacity rate of growth; otherwise the latter is at least as high as the former.

[5] Frobenius was born in 1849, Markov in 1856, and Perron in 1880.

If $T > T^*$, capitalists work workers more than that required for reproduction of the labouring force and pay them only partly; workers are overworked, underpaid and, hence, exploited by capitalists.

Marx thus divided the total supply of labour by a worker, T, into a paid part T^* and an unpaid part $T - T^*$, both measured in terms of labour time; and he defined the rate of exploitation e as $(T - T^*)/T^*$. Using this definition, he established a theorem to the effect that the equilibrium rate of profit and the equilibrium rate of growth are positive if and only if the rate of exploitation e is positive. In proving this theorem (I call the part concerning the rate of profit the fundamental Marxian theorem), Marx often confused the two distinct accounting systems, one in terms of prices, the wage rate and the rate of profit, and the other in terms of values, the value of the labour power and the rate of exploitation. The former describes the equations which equilibrium prices and the equilibrium rate of profit established by competitive arbitrage must satisfy, while the latter provides the equations for calculating the quantities of labour needed to produce goods by the techniques actually prevailing in the economy. Evidently they should be strictly distinguished from each other, so that the theorem must be re-proved without any confusion. This has been done elsewhere with great care, confirming Marx's results with minor revisions; there is no need to reproduce the proof here.[6]

It is important to note that this proof requires the value calculation. However, as I have pointed out in [7, Ch. 14], a number of severe assumptions must be fulfilled in order to avoid the ambiguous cases of the values of commodities not being determined uniquely, as well as the meaningless cases of some commodities taking on negative values. These assumptions rule out inter alia joint production and choice of techniques, so that we cannot treat capital good i of age $t + 1$ as a joint output appearing at the end of the process which uses capital good i of age t. We also cannot treat the problem of the determination of the economic lifespans of capital goods as a choice problem concerning those alternative processes which use capital goods of different ages. Therefore, unlike von Neumann and like neoclassical economists, Marx assumed that capital goods are malleable and autonomously evaporate as in radioactive decay and, hence, his theory of reproduction cannot deal with the so-called capital age-structure problems that arise when capital goods depreciate continuously and are bodily replaced in a discrete way.

As soon as joint production and choice of techniques are admitted, we must discard the labour theory of value, at least in the form Marx formulated it. So if the concept of value is indispensable for the definition of exploitation, the fundamental Marxian theorem is not applicable in the general case of durable capital goods being treated in the von Neumann way. If this is true, it is obviously a great disaster from the point of view of Marxian economics. At the end of my book, I tried some explorations to rescue Marx. I found [7, pp. 179–196]: (i) There is an alternative way to formulate the labour theory of value, not as the theory of "actual values" calculating the embodied-labour contents of commodities on the basis of

[6] For the Fundamental Marxian Theorem, see [7, pp. 53–71]. Once it is established, the positiveness of the equilibrium rate of growth immediately follows, because this is equal to the equilibrium rate of profit as von Neumann proved. Also, see [**8**].

the prevailing production coefficients as Marx did, but as the theory of "optimum values" considering values as shadow prices determined by a linear programming problem that is dual to another linear programming problem for the efficient utilisation of labour.[7] (ii) Optimum values are not necessarily determined uniquely. But the rate of exploitation is well defined in terms of optimum values in spite of the existence of joint products and alternative methods of production, provided that heterogeneous labour does not exist.

However, in spite of these findings, I must admit that when I wrote the book, I did not know whether or not the fundamental Marxian theorem was valid in the general model with durable capital goods, so that the book unfortunately ends with an open question. I now want to restart the rescue operation. The point is whether the theorem can be re-established by reformulating it in terms of optimum values. But one thing must be settled before examining this possibility: that is, whether Marx would be prepared to accept our recommendation, if we can provide one.

In this respect I am optimistic. We remember that Marx gave three different definitions of the rate of exploitations: (i) the ratio of unpaid labour to paid labour, (ii) the ratio of surplus value to the value of labouring power, and (iii) the ratio of surplus labour to necessary labour. He also proved the identity of these three on the assumption that the values of commodities could be calculated in an unambiguous way [7, Ch. 5]. However, in the case of values not being determined uniquely, the first two definitions become unintelligible and useless. In fact, in the second definition the value of labouring power (i.e., the value of the subsistence-consumption bundle of commodities) and the value of surplus outputs are reckoned in terms of the values of individual commodities, while in the first definition payment to labour is measured in terms of values. The rate of exploitation defined either way may take on various numerical values, depending upon the particular value system chosen to calculate it. The only definition which remains well defined after the death of the concept of value (actual value), provided there is no heterogeneous labour, is the third one; as I said at the end of my book, the ratio of surplus labour to necessary labour is determined definitely, even though values may be uncertain or not positive because of the existence of joint outputs and alternative methods of production.

Let us assume that all labours are homogeneous. (Throughout the following, we ignore all the difficulties which arise from the existence of different kinds of labour. As Samuelson has appropriately pointed out, this is a simplification which is now made by many neo-classical economists.) Let N be the number of workers actually employed. Each worker works T hours a day and is paid wages at the subsistence level. We denote the subsistence-consumption vector (per man) by C so that N units of C are required to keep N workers alive. The "necessary labour" is defined as the minimum labour time necessary to produce consumption goods CN, while the "surplus labour" is the excess of the actually consumed labour time over the necessary labour time.

[7] Although actual values and optimum values should be clearly distinguished, we refer to actual value simply as value below until the concept of optimum value is introduced.

To calculate the minimum labour time required to produce CN, we must have information about all the available techniques of production, actually chosen or potentially usable. Let B be the output-coefficient matrix, A the physical-input coefficient matrix, and L the labour-input coefficient row vector.[8] Processes are defined in the von Neumann way. A sufficient number of fictitious commodities and a sufficient number of fictitious processes are introduced in order to standardise the production periods and the lifespans of capital goods, all equalling one period [**6**, pp. 89–114]. There is no reason why the number of available processes should equal the number of commodities, so that B and A are in general rectangular. Finally, x represents the column vector of the intensities of operation of the processes and is simply referred to as the operation vector.

The necessary labour, min Lx, is obtained by solving the linear programming problem:

(P.1) Minimise Lx subject to

$$(1) \qquad Bx \geqq Ax + CN, \qquad x \geqq 0.$$

Let x^0 be a solution; then min $Lx = Lx^0$. We assume:

ASSUMPTION 1: *Labour is indispensable to produce the consumption basket C; that is,*

$$(2) \qquad Lx^0 > 0.$$

It is noted that x^0 is not necessarily unique, but min Lx is of course determined uniquely. The rate of exploitation is then given as

$$(3) \qquad e = \frac{\text{surplus labour}}{\text{necessary labour}} = \frac{TN - Lx^0}{Lx^0}$$

which is uniquely determined. The optimum operation vector x^0 may be different from the actual one, x^a. First, in the actual capitalist economy, x^a may not be an equilibrium operation vector. Secondly, even if it is, those processes which are actually chosen in the state of equilibrium are processes whose rates of profit are the largest, but not those which minimise employment of labour. Thirdly, in x^a, unlike x^0, processes for the production of luxury goods and investment goods may be operated at positive intensities. As $TN = Lx^a$, we have $e = (Lx^a - Lx^0)/Lx^0$, so that e is zero in the exceptional case where $x^a = x^0$.

Next we consider a dual linear programming problem of (P.1). It is stated as:

(P.1*) Maximise ΛCN subject to

$$(4) \qquad \Lambda B \leqq \Lambda A + L, \qquad \Lambda \geqq 0.$$

Let Λ^0 be a solution to this problem; then max $\Lambda CN = \Lambda^0 CN$, and by the duality theorem we have

$$(5) \qquad \Lambda^0 CN = Lx^0.$$

[8] Column i of A or B lists input or output coefficients of process i, while row j those of commodity j. Of course A and B are non-negative and non-zero.

The vector of shadow prices Λ^0 gives what I called "optimum values" of commodities in [7]. As Λ^0 may not be unique, there may be many (in fact, infinitely many) optimum value systems. Substituting from (5), formula (3) may be put in the form:

$$e = \frac{T - \Lambda^0 C}{\Lambda^0 C}, \tag{6}$$

which gives the paid-unpaid definition of the rate of exploitation in terms of optimum values. Notice that in this definition the subsistence-consumption vector C is evaluated at optimum values instead of actual values, which Marx used to calculate e. Like actual values, optimum values may not be unique if joint outputs and alternative methods of production are admitted. But unlike actual values, they give a unique evaluation of C; that is to say, $\Lambda^0 C$ takes on the same value for all optimum value systems as (5) shows. It is worth mentioning that e of (6) as well as e of (3) is not a mysterious concept; it can be calculated objectively once data A, B, C, L, and N are given.

We now want to generalise the fundamental Marxian theorem. The problem is to find the necessary and sufficient condition for the capitalist economy to be profitable and capable of expansion. We begin by defining the profitability of the economy. Let p be the row vector of prices, w the hourly-wage rate, and π_i the rate of profit of process i. By definition, we have

$$(pB)_i = (1 + \pi_i)(pA + wL)_i \qquad \text{for all } i,$$

where $(X)_i$ represents the ith component of vector X. By defining $\pi =$ the largest of the π_i's, these equations can be put in the form of vector inequality:

$$pB \leqq (1 + \pi)(pA + wL),$$

that may then take the form:

$$pB \leqq (1 + \pi)p(A + DL), \tag{7}$$

where $D = C/T$, because the wage rate is set at the subsistence level so that $wT = pC$, or $w = pD$. D represents the subsistence-consumption per man-hour, and $A + DL$ is usually referred to as the matrix of augmented input coefficients.

If strict inequality holds for process i, then π_i is smaller than the maximum rate of profit π, so that capitalists will not adopt process i. Then the maximum rate of profit will prevail in the economy because of this competitive choice of techniques. However, the π depends on p. The rate of profit that is *guaranteed* by the given technology $(A, L; B)$ and the given subsistence-consumption per man-hour, D, is the minimum of π satisfying (7) with non-negative, non-zero p. We may refer to this minimum value of π, denoted by π^w, as the warranted rate of profit and a p associated with π^w as p^w.

Let us next determine the capacity growth rate of the economy. Evidently the capitalist economy can grow at its full capacity only when capitalists invest their entire income and workers are paid only the subsistence wages. Then there can be no capitalists' consumption, and the total demand for goods amounts to:

$$Ax_t + CN_t, \tag{8}$$

where x_t is the operation vector in period t and N_t the number of workers employed in period t. N_t equals the total amount of labour time required for operation x_t divided by the working hours per man per day, i.e., $N_t = Lx_t/T$. Substituting, (8) can be rewritten as $(A + DL)x_t$. The feasibility of production requires

$$(9) \qquad Bx_{t-1} \geqq (A + DL)x_t,$$

because the period of production is standardised to equal 1.

Now let g_i be the rate of increase in the intensity of operation of process i. The rate of growth of the system is determined by the minimum of the rates of growth of individual processes, the g_i's. We have from (9)

$$(10) \qquad Bx \geqslant (1 + g)(A + DL)x,$$

where g = the smallest of the g_i's, and subscript $t - 1$ is omitted from both sides. We may now measure the growth capacity of the economy by the maximum balanced-growth rate that is obtained by maximising g subject to (10) with non-negative, non-zero x. We denote the capacity growth rate, max g, by g^c and an x associated with g^c by x^c.

We now establish the relationship of the rate of exploitation e, (i) to the warranted rate of profit π^w and (ii) to the capacity growth rate of the economy g^c. For this purpose we make the following two assumptions:

ASSUMPTION 2: *When workers are paid no wages, capitalists are guaranteed positive profits; that is to say, the minimum value of π satisfying $pB \leqq (1 + \pi)pA$ with non-negative, non-zero p is positive.*[9]

ASSUMPTION 3. *Labour is indispensable for the economy to grow at the capacity growth rate; that is to say, $Lx^c > 0$.*

Using these assumptions, we can prove:

LEMMA 1: *That the rate of exploitation is positive ($e > 0$) implies that the warranted rate of profit is positive ($\pi^w > 0$).*

PROOF: When $p^wD = 0$, inequality (7) holding with π^w and p^w, i.e., inequality

$$(11) \qquad p^wB \leqq (1 + \pi^w)p^w(A + DL)$$

is reduced to

$$p^wB \leqq (1 + \pi^w)p^wA,$$

so that $\pi^w > 0$ by Assumption 2. Thus the implication asserted by Lemma 1 is trivial in this case.

Next we prove the lemma for the case of $p^wD > 0$. First, in view of (3) and the definition, $D = C/T$, we have from (1)

$$Bx^0 \geqq Ax^0 + DLx^0(1 + e).$$

[9] This assumption is basic to all economic analyses. If it is not satisfied, positive profits cannot occur so that no capitalist economy exists.

Pre-multiply this by the non-negative, non-zero vector p^w. Secondly, post-multiply (11) by x^0. We then obtain

$$p^w A x^0 + (1 + e) p^w D L x^0 \leqq p^w B x^0 \leqq (1 + \pi^w)(p^w A x^0 + p^w D L x^0).$$

Hence,

$$e p^w D L x^0 \leqq \pi^w (p^w A x^0 + p^w D L x^0).$$

In this expression $Lx^0 > 0$ and $p^w D > 0$ by assumption, and $p^w A x^0 \geqq 0$. Therefore, $e > 0$ implies $\pi^w > 0$.

LEMMA 2: *That the capacity growth rate of the economy is positive* ($g^c > 0$) *implies that the rate of exploitation is positive* ($e > 0$).

PROOF: By definition, g^c and x^c satisfy

$$B x^c \geqq (1 + g^c)(A + DL) x^c.$$

Pre-multiplying this by Λ^0 and post-multiplying

$$\Lambda^0 B \leqq \Lambda^0 A + L$$

by x^c, we have

$$(1 + g^c)\Lambda^0 (A + DL) x^c \leqq \Lambda^0 B x^c \leqq \Lambda^0 A x^c + L x^c,$$

so that

$$g^c \Lambda^0 (A + DL) x^c \leqq L x^c - \Lambda^0 D L x^c. \tag{12}$$

As we have

$$\Lambda^0 D L x^c = \frac{\Lambda^0 C}{T} L x^c = \frac{L x^0}{TN} L x^c$$

by (5), the right-hand side of (12) can be written as

$$\left(1 - \frac{L x^0}{TN}\right) L x^c.$$

Therefore, in view of the definition of e, we have from (12)

$$g^c \Lambda^0 (A + DL) x^c \leqq e \frac{L x^0}{TN} L x^c.$$

In this expression, $Lx^c > 0$ by Assumption 3 and $\Lambda^0 D > 0$ by (2) and (5). Hence $g^c > 0$ implies $e > 0$. *Q.E.D.*

Notice that Assumption 3 plays a crucial role in proving Lemma 2. If $Lx^c = 0$, that is to say, the economy grows at the capacity rate without using labour, then g^c may be positive, irrespective of the sign of e. However, in the case of perfect automation, the concept of exploitation is meaningless.

LEMMA 3: *The capacity growth rate of the economy g^c is at least as large as the warranted rate of growth π^w; that is, $g^c \geqq \pi^w$.*

This lemma has been proved by post-von Neumann economists [**6**, Appendix]. However, they were unaware of Lemmas 1 and 2 so that they could not obtain the necessary and sufficient condition for π^w and g^c to be positive. We can now at last prove:[10]

THEOREM 1 (The Generalised Fundamental Marxian Theorem): *Positive exploitation is necessary and sufficient for the system to have positive growth capacity as well as to guarantee capitalists positive profits. In other words, $\pi^w > 0$ and $g^c > 0$ if and only if $e > 0$.*

The proof is easy. First, by Lemma 1, $e > 0$ implies $\pi^w > 0$. Conversely, $\pi^w > 0$ implies $g^c > 0$ by Lemma 3, which in turn implies $e > 0$ by Lemma 2.

Secondly, $e > 0$ implies $\pi^w > 0$, as before, which in turn implies $g^c > 0$ by Lemma 3. Conversely, $g^c > 0$ implies $e > 0$ by Lemma 2.

Thus, under Assumptions 1, 2, and 3, three propositions (i) that capitalists exploit workers ($e > 0$), (ii) that the capitalist system is profitable ($\pi^w > 0$), and (iii) that the capitalist system is productive ($g^c > 0$) are all equivalent. Obviously the theorem is an extension of the fundamental Marxian theorem but differs in an important aspect from the original one. It does not require the concept of "actual values" because the rate of exploitation is defined in terms of the actual employment of labour TN and the minimum employment Lx^0 necessary to produce the commodities for subsistence, as is shown in formula (3); or in terms of optimum values Λ^0, as in formula (6). In spite of the possible non-uniqueness of the optimum value system, the theorem in the latter form is not ambiguous, because, as has been seen, the rates of exploitation calculated on the basis of different optimum value systems are all equal to the rate of exploitation in terms of the actual and the minimum

[10] Note that the theorem is proposed as a long-run proposition, so that it may be consistent with fluctuations in the rates of profit and the rates of growth of individual processes from period to period.

Secondly, the theorem may be further generalised in the following way. Let r be an arbitrary positive number. Let x^0 be a solution to the following problem: Minimise $(1 + r)Lx$ subject to

$$Bx \geqslant (1 + r)Ax + CN, \qquad x \geqq 0.$$

Define

$$e(r) = \frac{TN - (1 + r)Lx^0}{(1 + r)Lx^0}.$$

Using Assumptions 1 and 3, we can prove the following Lemma 2* and its converse:

LEMMA 2*: *$g^c > r$ implies $e(r) > 0$.*

Hence,

THEOREM 1*: *$g^c > r$ if and only if $e(r) > 0$.*

However, it must be noted that even though Assumption 2 is added, $e(r) > 0$ does not imply $\pi^w > r$. A sufficient condition for this result is that there is at least one consumption good which is not free at p^w; that is, $p^wD > 0$. This is a strong assumption. It, together with Assumption 3, implies that the system is indecomposable, so that $g^c = \pi^w$.

Also it can be shown that $e(r) > 0$ is sufficient but *not* necessary for $\pi^w > 0$, unless the system is indecomposable.

employment. Thus Marx will not die together with the labour theory of value (actual value) as long as the fundamental Marxian theorem is considered the core of his economic theory.

Finally, let us see that in the special case where the labour theory of value rigorously holds, i.e., the case of A being a square matrix and B the identity matrix, $A + DL$ satisfies the Hawkins-Simon condition if and only if the rate of exploitation e is positive. In this case, it can be shown that $1 + \pi^w$ equals the reciprocal of the Frobenius root (the largest eigenvalue) of $A + DL$. Condition $e > 0$ is necessary and sufficient for $\pi^w > 0$ and hence for the Frobenius root of $A + DL < 1$. Therefore the augmented input-coefficient matrix is productive and satisfies the Hawkins-Simon condition.

3. THE TRANSFORMATION PROBLEM

Another example of the new mathematics which Marx happened to meet in *Capital* may be found in his discussion of the so-called transformation problem. In this problem he was concerned with the correspondence between the long-run equilibrium prices (or production prices) of commodities as solutions to the price-determination equations and values (or embodied-labour contents) as solutions to the value-determination equations. Marx tacitly assumed that values (actual values) could be determined unambiguously. This means that each commodity is produced by one and only one process of production with no joint output. That is to say, Marx tackled the problem with two hidden assumptions: (i) no alternative processes so that the input-coefficient matrix A is square, and (ii) no joint production so that the output-coefficient B is the identity matrix I.

With these assumptions, the value-determination and the price-determination equations are written as

$$\Lambda = \Lambda A + L \tag{13}$$

and

$$p = (1 + \pi)(pA + wL), \tag{14}$$

respectively, where Λ denotes the value vector, p the price vector, and π the equilibrium rate of profit. Taking into account the budget equation of the worker, $w = pD$, we may rewrite (14) in the form:

$$\bar{p} = (1 + \bar{\pi})\bar{p}(A + DL), \tag{15}$$

where D represents the vector of subsistence-consumption per man per hour and $A + DL$ the augmented input-coefficient matrix. Equation (15) implies that $1 + \bar{\pi}$ is determined as the reciprocal of an eigenvalue of $A + DL$ and $\bar{p}$ as the associated row eigenvector. Moreover, the conditions that $1 + \bar{\pi}$ be positive and $\bar{p}$ be non-negative, non-zero should be imposed because they are interpreted as 1 plus the rate of profit, and the price vector, respectively.[11] Are there such $1 + \bar{\pi}$ and $\bar{p}$? If so, how can they be obtained?

[11] As has been seen, the fundamental Marxian theorem establishes the positiveness of $\bar{\pi}$.

Let us now put

$$(16) \qquad M = \Lambda + DL,$$

which is a non-negative, square matrix. Then we see that Marx's problem is exactly the same as the one which Frobenius, Perron, and Markov met later. That is to say, does a non-negative square matrix M have a non-negative, non-zero eigenvector that is associated with a positive eigenvalue? To our contemporaries this question is no more than a simple, bookish question asked about Frobenius' theorem on non-negative matrices. But it was an entirely new mathematical problem when Marx was writing *Capital*.

It is true that Marx was perplexed and confused. But it is also true that he groped for the solution and almost got it. Unless we are inspired with antiquarian interests, however, it is senseless to reproduce his argument exactly because it suffered from confusions. We must make some minimum corrections and modifications to his argument to find out what may deserve to be called Marx's solution. In my book, I proposed an interpretation of Marx which boils down to the following two equations:[12]

$$(17) \qquad \bar{\pi} = e\frac{\Lambda DL\bar{y}}{\Lambda M\bar{y}} = e\frac{V\bar{y}}{(C + V)\bar{y}},$$

$$(18) \qquad p_t = (1 + \bar{\pi})p_{t-1}M,$$

where $\bar{y}$ is the column eigenvector associated with the largest positive eigenvalue of M that is shown to be equal to the long-run equilibrium balanced-growth output vector (or the von Neumann equilibrium output vector), V the vector of variable capitals ΛDL, and C the vector of constant capitals ΛA.[13]

[12] For bibliographical evidence of this interpretation, see [7, pp. 56–86]. Equation (17) is obtained in the following way: By definition of the rate of exploitation, we have

$$(1 + e)\Lambda D = 1,$$

so that we may write the value equation (13) as

$$\Lambda = \Lambda A + \Lambda DL + e\Lambda DL$$

which expresses value = constant capital + variable capital + surplus value. Post-multiplying this by $\bar{y}$ and taking the definition of M into account, we have

$$\Lambda y - \Lambda M\bar{y} = e\Lambda DL\bar{y}.$$

On the other hand, by the definition of $\bar{y}$, we have

$$\bar{y} = (1 + \bar{\pi})M\bar{y},$$

where $1 + \bar{\pi}$ is the reciprocal of the largest positive eigenvalue of M. Pre-multiply the above equation by Λ. Then

$$\Lambda\bar{y} - \Lambda M\bar{y} = \bar{\pi}\Lambda M\bar{y}.$$

Hence, $\bar{\pi}\Lambda M\bar{y} = e\Lambda DL\bar{y}$, so that we have (17).

[13] In the previous section, C denoted the subsistence-consumption vector. In the following, it is used to represent the vector of constant capital.

Formula (17) transforms the rate of exploitation e into the rate of profit $\bar{\pi}$. In Marx's original formula, the "unit" vector, that is, a column vector with components being all unity, replaces the eigenvector $\bar{y}$. But in that case, e is not correctly transformed into $\bar{\pi}$ so that the revision is necessary. With given A, L, and D, we can calculate Λ and, hence, C, V, and e. We can also calculate $\bar{y}$, as M is known. Therefore, by (17) we obtain the $\bar{\pi}$ which corresponds to the largest positive eigenvalue of M.

On the other hand, (18) is the formula used to transform values into prices. In (18) we may regard the matrix $M^* = (1 + \bar{\pi})M$ as given because M is given and $\bar{\pi}$ is determined by (17). It is easily seen that M^* is a Markov matrix, that is, a non-negative matrix whose largest positive eigenvalue is unity. Hence, provided that M^* is primitive,[14] the ergodic solution to (18), or the eigenvector of M* that is associated with the largest eigenvalue 1, is obtained as the limit of that infinite sequence $p_0, p_1, \ldots, p_t, \ldots$, which starts with the arbitrary non-negative, non-zero vector p_0 and is generated in a recursive way according to formula (18). This iteration method for finding the long-run equilibrium price vector $\bar{p}$ will of course be most effective if the initial point p_0 is taken very near to $\bar{p}$.

Is there then any point which we may safely assume to be near the equilibrium $\bar{p}$? Marx started the sequence at $p_0 = \Lambda$, because values would give the long-run equilibrium prices in the society of "simple commodity production" (as the classical labour theory of value claims) and so these values would not be very far from the corresponding equilibrium prices in the capitalist society, though some deviations of the equilibrium prices from the values are inevitable, unless each and every industry in the economy has the same value composition of capital. Thus to Marx, the iteration process (18) was a process of transforming the initial Λ into the ergodic $\bar{p}$. At one end of the Marxian transformation we have the long-run equilibrium price set of the classless "simple-commodity-production" economy and at the other the long-run equilibrium price set of the capitalist economy. Comparing them we can analyse the effects on the long-run equilibrium prices of commodities of a change in the social structure from one type of economy to the other.

Although I still believe that the above is an adequate mathematical formulation of Marx's transformation procedures, it is incomplete as an algorithm. It assumes that the column eigenvector $\bar{y}$ is known. Marx never asked how $\bar{y}$ is determined. In fact, he even had no idea of $\bar{y}$. Therefore, we take it as known and determine $\bar{\pi}$ and then $\bar{p}$. However, it is really a kind of circular reasoning to obtain the row eigenvector associated with an eigenvalue of a matrix on the assumption that the corresponding column vector is known. To get the column eigenvector (that is, the row eigenvector of the transposed matrix), we must assume, if no algorithm is available, that the row eigenvector (that is, the column eigenvector of the transposed matrix) is known. To avoid this kind of circularity we now propose to replace (17) by the iteration process:

$$(19) \qquad y_t = \frac{\Lambda y_{t-1}}{\Lambda M y_{t-1}} M y_{t-1}.$$

[14] Perhaps Marx implicitly assumed that M^* was "primitive." In fact, it is, under the plausible assumption that labour is indispensable, so that $L > 0$. For the definition of primitiveness and indecomposability (introduced later) of a matrix, see for example [5, pp. 14 and 163].

Let us now show that the sequence $\{y_t\}$ thus produced converges to the column eigenvector $\bar{y}$ of M corresponding to its largest eigenvalue.[15] For this purpose we explicitly assume the following.

ASSUMPTION 4: *The augmented input coefficient matrix M is primitive and indecomposable.*

Then we get the following two lemmas:

LEMMA 4: *The infinite sequence $\{y_t\}$ generated from an arbitrary non-negative, non-zero y_0 by* (19) *eventually converges to some non-negative, non-zero $\bar{y}$.*

PROOF: It is seen from (19) and Assumption 4 that $y_t \geqq 0, \neq 0$, for all t as $y_0 \geqq 0, \neq 0$. Also, from (19) $\Lambda y_t = \Lambda y_0 > 0$ for all t. Because $\Lambda > 0$,[16] this means that y_t is bounded. Therefore, by the Weierstrass theorem the sequence $\{y_t\}$ has at least one limiting point, say $\bar{y}_0$.

Consider now a sequence $\{\bar{y}_t\}$ starting from $\bar{y}_0$:

$$\bar{y}_t = \frac{\Lambda \bar{y}_{t-1}}{\Lambda M \bar{y}_{t-1}} M \bar{y}_{t-1}.$$

As $\bar{y}_0$ is a limiting point, $\bar{y}_t$ must sooner or later return to $\bar{y}_0$. Hence, there is an integer r such that

$$\bar{y}_1 = \frac{\Lambda \bar{y}_0}{\Lambda M \bar{y}_0} M \bar{y}_0,$$

$$\bar{y}_2 = \frac{\Lambda \bar{y}_1}{\Lambda M \bar{y}_1} M \bar{y}_1, \tag{20}$$

$$\vdots$$

$$\bar{y}_0 = \frac{\Lambda \bar{y}_{r-1}}{\Lambda M \bar{y}_{r-1}} M \bar{y}_{r-1},$$

where $\bar{y}_t \neq \bar{y}_0$, $t = 1, 2, \ldots, r-1$. If $r = 1$, we have:

$$\bar{y}_0 = \frac{\Lambda \bar{y}_0}{\Lambda M \bar{y}_0} M \bar{y}_0. \tag{21}$$

[15] Putting $H(y_{t-1}) = (\Lambda y_{t-1}/\Lambda M y_{t-1}) M y_{t-1}$, we may write (19) as $y_t = H(y_{t-1})$. It is then seen that the eigenvector $\bar{y}$ is a fixed point of the mapping $H(\cdot)$ into itself. Our problem is to prove the stability of $\bar{y}$. Solow and Samuelson were concerned with a similar problem. But we cannot apply their theorems, because their basic assumption that $\partial H/\partial y_{t-1} \geqq 0$ is not fulfilled in our case [**10**].

[16] Productiveness of A and indecomposability of M imply $\Lambda > 0$. As A is productive and L non-negative, non-zero, it follows from $\Lambda = \Lambda A + L$ that Λ is non-negative, non-zero. Suppose Λ is not strictly positive, i.e., it can be partitioned into (Λ_1, Λ_2) such that $\Lambda_1 > 0$, $\Lambda_2 = 0$. Let

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

be the corresponding partition of M. Indecomposability of M implies $M_{12} \neq 0$. That is, if $A_{12} = 0$, $L_2 \neq 0$, while if $L_2 = 0$, $A_{12} \neq 0$. Therefore, in view of $\Lambda_2 = \Lambda_1 A_{12} + L_2$ and $\Lambda_1 > 0$, we find that $\Lambda_2 \neq 0$, a contradiction. Hence $\Lambda > 0$.

But if $r > 1$, we have from (20):

$$\lambda\bar{y}_0 = M^r\bar{y}_0, \qquad \lambda\bar{y}_1 = M^r\bar{y}_1, \qquad \ldots, \qquad \lambda\bar{y}_{r-1} = M^r\bar{y}_{r-1}, \tag{22}$$

where λ is the reciprocal of

$$\prod_{s=0}^{r-1} \frac{\Lambda\bar{y}_t}{\Lambda M\bar{y}_t}.$$

Equation (22) implies that λ is an eigenvalue of M^r and r different (not proportional) eigenvectors $\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{r-1}$ are associated with the same λ. This means that M is not primitive.[17] This contradicts Assumption 4; therefore, (21) must hold.

Equation (21) implies that the sequence $\{y_t\}$ starting from y_0 stays within a sufficiently small neighbourhood of $\bar{y}_0$ once it gets close enough to $\bar{y}_0$. This establishes the convergence of $\{y_t\}$ to $\bar{y}_0$.

LEMMA 5: *Let $\bar{y}_0$ be any non-negative, non-zero vector satisfying* (21). *Then $\Lambda\bar{y}_0/\Lambda M\bar{y}_0$ equals $1 + \bar{\pi}$, i.e., the reciprocal of the largest positive eigenvalue of M.*

PROOF: It is seen from (21) that $\bar{y}_0$ is an eigenvector of M and $\Lambda\bar{y}_0/\Lambda M\bar{y}_0$ is the reciprocal of the corresponding eigenvalue. Suppose the contrary of the lemma, so that $\Lambda\bar{y}_0/\Lambda M\bar{y}_0 = 1 + \pi_0 \neq 1 + \bar{\pi}$. Let $\bar{p}$ be the row eigenvector of M corresponding to $1 + \bar{\pi}$. Then we have

$$\Lambda\bar{y}_0 = (1 + \pi_0)M\bar{y}_0 \qquad \text{and} \qquad \bar{p} = (1 + \bar{\pi})\bar{p}M.$$

Therefore,

$$\bar{p}\bar{y}_0 = (1 + \pi_0)\bar{p}M\bar{y}_0 = (1 + \bar{\pi})\bar{p}M\bar{y}_0.$$

As $\bar{p}$ is shown to be positive[18] and $\bar{y}_0$ is non-negative, non-zero, we have $\bar{p}\bar{y}_0 > 0$, so that $1 + \pi_0 = 1 + \bar{\pi}$, a contradiction.

THEOREM 2: *The infinite sequence generated by* (19) *from an arbitrary initial point y_0 which is non-negative and non-zero converges to the column eigenvector $\bar{y}$ of M which is associated with the largest eigenvalue of M.*

PROOF: It is clear from Lemmas 4 and 5 that the sequence $\{y_t\}$ converges to a non-negative, non-zero $\bar{y}_0$ at which

$$\bar{y}_0 = (1 + \bar{\pi})M\bar{y}_0.$$

Hence $\bar{y}_0$ is the eigenvector of M associated with its largest eigenvalue, $1/(1 + \bar{\pi})$.

Q.E.D.

Formulas (18) and (19) enable us to get rid of the circular reasoning already mentioned. The complete algorithm works in the following way. First, starting

[17] Suppose $\bar{y}_0$ and $\bar{y}_1$ are not proportional, so that $\min (\bar{y}_0)_i/(\bar{y}_1)_i$ (denoted by h) is different from $\max (\bar{y}_0)_i/(\bar{y}_1)_i$. Therefore $z = y_0 - hy_1$ is a non-negative, non-zero vector with $(z)_i = 0$ for some i. From (22), $\lambda z = M^r z$, which implies $\lambda^m z = M^{mr} z$ for any integer m. On the other hand, as M is primitive and indecomposable, we have $M^k > 0$ for large k. Therefore $M^{mr} > 0$ for large m. Hence $z > 0$, a contradiction.

[18] This follows the indecomposability of M.

with an arbitrary non-negative, non-zero vector y_0 and calculating y_1, y_2, and so on successively according to (19), we finally obtain a stationary solution $\bar{y}$. Secondly, compute $\Lambda\bar{y}/\Lambda M\bar{y}$ which gives $1 + \bar{\pi}$. As $\Lambda\bar{y}/\Lambda M\bar{y} = 1 + eV\bar{y}/\Lambda M\bar{y}$, it is at once seen that the rate of exploitation e is transformed into the rate of profit $\bar{\pi}$ in exactly the same way as in formula (17). Thirdly, substituting the $1 + \bar{\pi}$ thus obtained into (18), we can calculate the sequence $\{p_t\}$ beginning with the value vector Λ by taking it as the initial price vector. The sequence $\{p_t\}$ converges and the limit gives the long-run equilibrium price set.

Marx obtained a number of conclusions concerning the transformation of the value accounting system into the price accounting system. The five most important are discussed in my *Marx* [7, pp. 72–73]. His conclusions (iv) and (v) concerning the relationship among "production price," "value," and "value composition of capital" require some revisions [7, pp. 81–84], but they are not as important, from Marx's point of view, as the first three conclusions. These state: (i) The sum of the prices of production of all commodities equals the sum of their values; (ii) the cost-price of a commodity is smaller than its value; and (iii) the total surplus value equals the total profits. In my book I have said that these conclusions are true only if industries are "linearly dependent."[19] However, I now find that whereas this accusation is right regarding conclusion (ii), it is wrong with respect to the other two. As will be seen below, the truth of conclusions (i) and (iii) is independent of the condition of linear dependence of industries, provided that the economy is in the state of long-run equilibrium balanced growth (or the von Neumann equilibrium), so that commodities are produced in proportion to the eigenvector $\bar{y}$. And it is these conclusions which Marx would never want to deny.

THEOREM 3: *Let $\bar{p}$ be the long-run equilibrium price vector obtained by iteration* (18) *with $p_0 = \Lambda$, and $\bar{y}$ the long-run equilibrium balanced-growth output vector. When production is made at $\bar{y}$, the aggregate output in terms of prices $\bar{p}$, i.e., $\bar{p}\bar{y}$, equals the aggregate output in terms of values Λ, i.e., $\Lambda\bar{y}$. Also the total profits $\Pi\bar{y}$ equals the total surplus value $S\bar{y}$, where Π represents the vector of profits per unit of output and S the vector of surplus values per unit of output.*

PROOF: It is not difficult to prove the theorem. Post-multiplying (18) by $\bar{y}$, we have:

$$(23) \qquad p_t\bar{y} = (1 + \bar{\pi})p_{t-1}M\bar{y}.$$

On the other hand, as $\bar{y}$ is the column eigenvector of M associated with the eigenvalue, $1/(1 + \bar{\pi})$, we have:

$$\bar{y} = (1 + \bar{\pi})M\bar{y},$$

so that $1 + \bar{\pi} = p_{t-1}\bar{y}/p_{t-1}M\bar{y}$. Substituting from this, (23) yields $p_t\bar{y} = p_{t-1}\bar{y}$ which holds for all t. Noticing that $p_0 = \Lambda$ and $\bar{p} = \lim_{t\to\infty} p_t$, we get:

$$(24) \qquad \Lambda\bar{y} = \bar{p}\bar{y}.$$

[19] For the definition of the linear dependence of industries, see [7, pp. 77–78].

Next, since $p_1\bar{y} = (1 + \bar{\pi})\Lambda M\bar{y}$, $p_{t+1}\bar{y} = (1 + \bar{\pi})p_t M\bar{y}$, and $p_1\bar{y} = p_t\bar{y}$ for all t, we get:

$$\Lambda M\bar{y} = p_t M\bar{y} \qquad \text{for all } t,$$

so that

$$\Lambda M\bar{y} = \bar{p}M\bar{y}.$$

Subtracting this from (24), we get:

$$(25) \qquad S\bar{y} = \Pi\bar{y},$$

because $S = \Lambda - \Lambda M$ and $\Pi = \bar{p} - \bar{p}M$. *Q.E.D.*[20]

Obviously this result is very favourable to Marx, although it is not exactly what he asserted. The iteration process (18) assures that, as long as the sequence $\{p_t\}$ starts from $p_0 = \Lambda$, it converges to the long-run equilibrium price vector at that particular absolute level at which both conditions, "total value equals total price" and "total surplus value equals total profit" are consistently satisfied. However, it should be noted that the simultaneous fulfilment of these two different "normalisation" conditions is subject to another qualification, that the aggregation is made at the long-run equilibrium balanced-growth output $\bar{y}$. When it is made at the actual production point x which may be different from $\bar{y}$, the sequence (18) may violate one of the two conditions or (probably) both.

Unless some stringent assumptions are made, it is impossible to get an iteration formula which produces a sequence of price vectors that simultaneously satisfies, in the limit, the two normalisation conditions,

$$(26) \qquad \Lambda x = \bar{p}x$$

[20] Just before the publication of my book, Professor Okishio proposed an interpretation of Marx's algorithm of the transformation of values into

$$(18') \qquad p_t = \frac{p_{t-1}x}{p_{t-1}Mx}p_{t-1}M,$$

where x is an arbitrary positive vector. He has shown that the sequence $\{p_t\}$ generated by the above formula from the initial point $p_0 = \Lambda$ converges to the long-run equilibrium price set $\bar{p}$ and $p_t x/p_t Mx$ converges to 1 plus the long-run equilibrium rate of profit. (But he missed the crucial condition that M must be primitive.) He has also shown that $\Lambda x = p_t x$ for all t and therefore $\Lambda x = px$. This result is stronger than my (24), because it holds for any x, but Okishio's sequence does not satisfy the other normalisation condition, $Sx = \Pi x$, for general x. Obviously, Okishio's formula (18′) is reduced to (18) when $x = \bar{y}$. I take $x = \bar{y}$ as Marx's hidden assumptions, because under it both conclusions (i) and (ii) of Marx become consistent.

Okishio has not proposed any algorithm for $\bar{y}$. But as $1 + \bar{\pi}$ is determined so as to equal $\lim_{t\to\infty} p_t x/p_t Mx$, we can calculate y_t according to

$$(19') \qquad y_t = (1 + \bar{\pi})My_{t-1};$$

then the sequence $\{y_t\}$ will converge to $\bar{y}$, provided y_0 is non-negative, non-zero. It is interesting to see that the system consisting of (18′) and (19′) is a dual to that of (18) and (19); x and p_t in (18′) play the roles of Λ and y_t in (19), respectively, while y_t in (19′) plays the role of p_t in (18). See [**9**].

and

(27) $$Sx = \Pi x,$$

for the actual output vector x which is not necessarily equal to the equilibrium output vector $\bar{y}$. But it is not difficult to obtain a formula which satisfies one of the two conditions. As has been seen in footnote (20), Okishio's formula (18′) is one such example. His sequence fulfills (26) (i.e., Marx's conclusion (i)), but not (27) (i.e., conclusion (iii)) in the limit. We may easily give another example. Let x be an output vector which is able to reproduce itself with surplus output of each commodity; in other words, x is a positive vector such that

(28) $$x > Mx.$$

There exist such x's because the rate of exploitation is positive so that M is "productive" by the fundamental Marxian theorem. Consider the sequence of non-negative vectors $\{p_t\}$ generated by

(29) $$p_t = \left(1 + \frac{Sx}{p_{t-1}Mx}\right)p_{t-1}M.$$

Post-multiply this by x. We obtain

(30) $$\begin{aligned} p_t x &= p_{t-1}Mx + Sx \\ &\leqslant kp_{t-1}x + Sx, \end{aligned}$$

where

$$k = \max_i k_i$$

and k_i represents the ratio of the ith component of Mx to the corresponding component of x. Because of (28), k is less than 1. Therefore we obtain from (30)[21]

(31) $$p_t x \leqslant \max\left(\frac{S_x}{1-k}, k\Lambda x + Sx\right) \qquad \text{for all } t.$$

This, together with the fact that p_t is non-negative, implies that p_t is bounded. Then Lemmas 4 and 5 apply mutatis mutandis, and we can show that the sequence $\{p_t\}$ generated by (29) converges to the row eigenvector $\bar{p}$ of M associated with the largest eigenvalue $1/(1 + \bar{\pi})$. Therefore, we see from (30) that

$$Sx = (\bar{p} - \bar{p}M)x = \Pi x$$

[21] Consider a difference equation

$$z_t = kz_{t-1} + Sx.$$

It is well known that its solution is monotonically increasing or decreasing and converges to $Sx/(1 - k)$, so that when z_0 is set at Λx,

$$z_t \leqslant \max\left(\frac{Sx}{1-k}, k\Lambda x + Sx\right) \qquad \text{for all } t.$$

As $p_t x \leqslant z_t$ from (30), we have (31).

holds in the limit, as long as the actual production vector x satisfies (28); but the other normalisation condition (26) is not generally satisfied. Thus the price vectors form a convergent sequence which is consistent with Marx's conclusion (iii) but not with (i).

Although it is very difficult to decide which one of the formulas, (18), (18′), (29) or any other, is closest to what Marx had in mind, the above argument should convince us that Marx met a very advanced mathematical problem in his economic studies. He was often confused, partly because of the difficulty of the problem and partly because of his lack of mathematical training, but it is really surprising to see that he almost succeeded in solving the problem.

Finally we are concerned with the traditional Marxian problem of the equal value composition of capital. It is well known that the value vector Λ is proportional to the production price vector $\bar{p}$ if and only if the value composition of capital is equal throughout all industries, i.e.,

$$\frac{C_1}{V_1} = \frac{C_2}{V_2} = \cdots = \frac{C_n}{V_n}.$$

It is easily seen that this condition is equivalent to

$$\frac{S\bar{y}}{(C + V)\bar{y}}(C + V) = S, \tag{32}$$

because $S = eV$ since the rate of exploitation is uniform throughout the economy.

However, it is evident that (32) is a very stringent condition, so that it is desirable to find a weaker condition which will extend the traditional result. In my book [7, pp. 76–80], I have shown that if and only if condition

$$\frac{S\bar{y}}{(C + V)\bar{y}}(C + V)M = SM, \qquad \text{or} \qquad \bar{\pi}(C + V)M^* = SM^*, \tag{33}$$

[22]

is satisfied, the second term p_1 of the Marxian sequence $\{p_t\}$ of (18), which starts from the first term $p_0 = \Lambda$, equals the true production price vector $\bar{p}$, so that there is no need to continue iteration (18) any further. I called (33) the condition of "linear dependence of industries" and p_1 "the Marxian price" because Marx calculated only p_1 in his numerical examples.

The condition of "linear dependence" is weaker than that of equal value composition of capital,[23] but it is true that it is still a very restrictive condition. In

[22] Note that $\bar{\pi} = S\bar{y}/(C + V)\bar{y}$ and $M^* = (1 + \bar{\pi})M$.

[23] Multiplication of (32) by M results in (33), so that when all industries have the same value composition of capital, they are linearly dependent. But the converse is not necessarily true, as the following numerical example shows. Let industries 1 and 2 produce capital goods and 3 consumption goods. Input coefficients are:

$$A = \begin{pmatrix} .1 & .2 & .3 \\ .7 & .4 & .1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad L = (20, 40, 60).$$

(continued on the next page)

the rest of this paper, therefore, we try to weaken condition (33) to obtain a general result.

Let us assume

$$\bar{\pi}(C + V)M^{*^u} = SM^{*^u} \tag{34}$$

with u being zero or some positive integer. Condition (34) for any $u = 1, 2, \ldots$ is weaker than the corresponding condition for $u - 1$.[24] It is at once seen that (34) reduces to the condition of linear dependence (33) when $u = 1$ and to the condition of equal value composition of capital (32) when $u = 0$, so that the following theorem holding for any $u = 0, 1, 2, \ldots$ is a generalisation of my previous result for the "linear dependence" case. It is also a generalisation of the familiar result for the classical "identical capital composition" case.

THEOREM 4: *If and only if* (34) *is satisfied for* $u = 0$ *or for some integer* u, *the terms* p_r, $r \geqslant u$, *of the sequence* $\{p_t\}$ *generated by* (18) *from* $p_0 = \Lambda$ *equal their limit* $\bar{p}$.

The proof which I have given in my book for the case of $u = 1$ mutatis mutandis verifies the theorem for any u. First, *necessity*: As $p_0 = \Lambda$, we have from (18)

$$p_u = \Lambda M^{*^u}, \tag{35}$$

while by $\bar{p} = p_u$,

$$p_u = p_u M^*. \tag{36}$$

Substituting (35) into (36), we get

$$\Lambda M^* = \Lambda M^{*^{u+1}}. \tag{37}$$

As $\Lambda = C + V + S$ and $\Lambda M = C + V$, we may put (37) in the form:

$$(C + V + S)M^{*^u} = (1 + \bar{\pi})(C + V)M^{*^u} \tag{38}$$

which may be put in the form (34).

Sufficiency: When (34) holds, it is clear that we have (38) and hence (37). Therefore,

$$\Lambda M^{*^u} = \Lambda M^{*^u} M^*,$$

We then have the value vector $\Lambda = (100, 100, 100)$. Let the vector of subsistence consumption per man-hour be

$$D' = (0, 0, .005),$$

where the prime denotes the transposition of vector D. The rate of exploitation e and the rate of profit $\bar{\pi}$ are then 1 and $\frac{1}{4}$, respectively. These data imply:

$$C = (80, 60, 40), \qquad V = (10, 20, 30), \qquad S = (10, 20, 30), \quad \text{and} \quad y' = (1, 2, 1).$$

Hence industries apparently differ in the value composition of capital. However, in view of the definition, $M = A + DL$, we have

$$\bar{\pi}(C + V)M = (18, 16, 14) \qquad \text{and} \qquad SM = (18, 16, 14),$$

so that the industries are "linearly dependent."

[24] Multiplying (34) for $u - 1$ by M^*, we obtain (34) for u but not vice versa.

which implies that (35) is the eigenvector of M^* associated with its eigenvalue 1. Thus $p_u = \bar{p}$. Once we have this, we at once find $p_r = \bar{p}$ for all $r > u$.

We now see that any primitive M^* approximately satisfies condition (34) for large u, provided it produces $\bar{\pi} > 0$. Let ε_u be the difference between $\bar{\pi}(C + V)M^{*^u}$ and SM^{*^u}; then

$$(39) \qquad \bar{\pi}(C + V)M^{*^u} = SM^{*^u} + \varepsilon_u.$$

Considering $C + V + S = \Lambda$, $C + V = \Lambda M$, and $M^* = (1 + \bar{\pi})M$ by definition, we have from (39)

$$(40) \qquad \Lambda M^{*^{u+1}} = \Lambda M^{*^u} + \varepsilon_u.$$

Since M^* is a primitive Markov matrix, M^{*^u} converges. Hence, from (40) $\lim_{u\to\infty} \varepsilon_u = 0$, which implies that any M^* approximately satisfies (34) if u is taken sufficiently large, and the p_u calculated by the Marxian transformation formula (18) approximates the ergodic solution $\bar{p}$ of the Markov matrix M^*.

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