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AGGREGATION IN LEONTIEF MATRICES AND THE LABOUR THEORY OF VALUE

BY M. MORISHIMA AND F. SETON

This article examines the relationship between the price of commodities and ther total labour content when the productive contributions of all other factors are imputed to labour. The transformation of one basis of valuation into the other, which entails the use of Leontief matrices, may be distorted by errors of aggregation. We examine the probable limits of these errors and the conditions for their total absence.

SOVIET AND Eastern European economists have recently been troubled by a number of questions which might be brought under a common heading as the "inverse transformation problem." After allowing an entirely artificial price system to emerge from uncoordinated *ad hoc* decrees, shifting fiscal and administrative requirements, or simple historical accident, they are increasingly feeling the lack of rational economic criteria for investment choice, import policy, modernization measures, and similar decisions requiring some objective balancing of economic advantage against economic cost. In the search for such criteria their long-standing doctrinal preconceptions have naturally tended to drive them back to the Marxian value concept which, though constructed without regard for scarcity relations, does at least afford some objective (i.e., non-arbitrary) and consistent basis for attaching relative economic weights to the heterogeneous products which must enter their calculations. The initial problem to be faced has therefore been the conversion of prices into "values" (direct and indirect labour absorbed per unit of a commodity).

The classical problem of the *opposite* conversion (values into prices) has exercised many minds since the appearance of the third volume of *Das Kapital*,[1] but the problem at issue here has so far only received attention as a by-product of the analysis of Leontief matrices. It has been shown,[2] for instance, that in a Leontief model the price of a commodity in terms of labour ("wage-price") will equal its Marxian "value" under certain conditions which include (i) competitive long-run equilibrium, i.e., zero profits in each sector, and (ii) perfect divisibility of the economy into "primitive sectors,"

[1] Böhm-Bawerk, Bortkievicz, Winternitz, Sweezy, May, Dobb, Meek, etc. (for exact references see Ronald Meek, "Some Notes on the 'Transformation Problem'," *Economic Journal*, March, 1956, p. 94). See also F. Seton ,"The Transformation Problem," *The Review of Economic Studies*, June, 1957, pp. 149–160.

[2] N. Georgescu-Roegen, "Leontief's System in the Light of Recent Results," *Review of Economics and Statistics*, Vol. 32, 1950, p. 217, and B. Cameron, "The Labour Theory of Value in Leontief Models," *Economic Journal*, Vol. LXII, 1952, pp. 191–7.

i.e., sectors producing single homogeneous commodities. Assumption (i) reduces the discovered equality to a merely formal one, since the destruction of the "surplus" (i.e., zero profit) does away with the substance of the Marxian analysis.

It is our purpose in this article to inquire into the general relationship between Leontief price[3] and Marxian value when both assumptions (i) and (ii) are relaxed. In the first section we shall allow nonzero profits, but retain the postulate of perfect divisibility. This is the exact obverse of the model examined by one of the authors in a recent article.[4] Cameron's model[5] will here emerge as a special case.

In the second section we shall relax the postulate of perfect divisibility, and inquire into the possible distorting effects of the inevitable sector aggregations underlying all statistically knowable Leontief tables.

Finally, in the third section, we shall apply the result of the general aggregation analysis to the special Marxian model characterized by the extreme form of aggregation into two departments only (capital goods and consumer goods).

1

Definitions

Consider an economy of n "primitive sectors" $i\,(i = 1, 2, \ldots, n)$ producing homogeneous outputs w_i (measured in terms of current prices), and let w_{ij} be the portion of sector i's output used up in the production of w_j. Writing v_i and s_i for the wage- and non-wage-factor incomes earned in the ith sector, and f_i for its final output (consumption plus investment), the basic equations of the familiar Leontief-type analysis take the form:

$$
\begin{array}{lllll}
w_{11} & + w_{12} & + \ldots + w_{1n} & + f_1 & = w_1 , \\
w_{21} & + w_{22} & + \ldots + w_{2n} & + f_2 & = w_2 , \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
w_{n1} & + w_{n2} & + \ldots + w_{nn} & + f_n & = w_n , \\
v_1 & + v_2 & + \ldots + v_n & & = v_0 , \\
s_1 & + s_2 & + \ldots + s_n & & = s_0 ,
\end{array} \tag{1}
$$

where the sum of any column i is by definition equal to w_i and $\Sigma f = \Sigma\,(v + s)$.

[3] It should be noted that Leontief price will in general differ from the Marxian "price of production" which posits a uniform rate of profit (on costs) in all sectors. This does not, however, affect the formal characteristics of the "inverse transformation problem," as the profit structure implicit in the price system need not be specified at all.

[4] F. Seton, *op. cit.*

[5] B. Cameron, *op. cit.*

Equations (2) represent the identical system in terms of Marxian "value" (instead of current price):

$$
\begin{array}{llllll}
\omega_{11} + \omega_{12} + \ldots + \omega_{1n} + \phi_1 & = \omega_1 \,, \\
\omega_{21} + \omega_{22} + \ldots + \omega_{2n} + \phi_2 & = \omega_2 \,, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\
\omega_{n1} + \omega_{n2} + \ldots + \omega_{nn} + \phi_n & = \omega_n \,, \\
\upsilon_1 \;\; + \upsilon_2 \;\; + \ldots + \upsilon_n & = \upsilon_0 \,, \\
\sigma_1 \;\; + \sigma_2 \;\; + \ldots + \sigma_n & = \sigma_0 \,.
\end{array} \tag{2}
$$

(The reader should note carefully throughout this article the distinction between the italic v and the Greek υ.)

The columns of (2) reproduce the familiar Marxian division of output into constant capital, variable capital, and surplus ($\omega_j = \sum \omega_{ij} + \upsilon_j + \sigma_j$). The first two elements may be bracketed together as the "total capital" (i.e., cost of production) $\varkappa$. In what follows we shall refer to (1) and (2) as the "price-system" and the "value-system" respectively. As far as possible we shall also retain the convention of denoting the value-transform of x by the corresponding Greek letter ξ.

It used to be held that the price-system was the only one which could be statistically known, and that the value-system was a metaphysical construct impervious to measurement or objective quantification of any sort. Our first task is to show that, given certain assumptions—all of them well grounded in the Marxian theory of value—this need not be so. Whatever the usefulness or irrelevance of the Marxian value concept as a description of "reality" or a guide to action, it is at least operationally meaningful.

We shall start by defining the (producers') "output quotas":

$$
c_{ij} \equiv \frac{w_{ij}}{w_i} = \frac{\omega_{ij}}{\omega_i} \equiv \gamma_{ij}; \quad \text{in matrix form: } c = \begin{pmatrix} c_{11} \ldots c_{1n} \\ \ldots\ldots\ldots \\ c_{n1} \ldots c_{nn} \end{pmatrix}. \tag{3}
$$

The advantage of using output quotas rather than the more popular input coefficients ($k_{ij} \equiv w_{ij}/w_j$) lies in the fact that the former are unaffected by the units of measurement used and will be identical in both price- and value-system.[6]

The vertical (by column) reading of (2) can then be put in matrix form:

$$
c'\omega + \upsilon + \sigma = \varkappa + \sigma = \omega \tag{4}
$$

[6] It is of course true that the degree of stability of output quotas in a *changing* economy is probably inferior to that of input coefficients, but this is of no consequence when we confine ourselves to recomputing the value base of *given* economies. Moreover, the quotas can easily be converted into input coefficients by the matrix formula $k = \hat{w}c\hat{w}^{-1}$ (where $\hat{w}$ is the diagonal matrix of outputs) and all theorems of this article can be restated in terms of k, if so desired.

where c' denotes the transposition of matrix c ($c'_{ij} = c_{ji}$) and the other symbols are column vectors of the elements indicated, e.g., $\omega = \{\omega_i\}$, $v = \{v_i\}$, etc.

Equation (4) cannot be solved (for ω) from a knowledge of the price system alone, since this supplies only c' and leaves v and σ undetermined, but it can be brought a step nearer solubility by the Marxian postulate that the "rate of exploitation" (σ_i/v_i) is equal in all sectors. This may be put as:

$$v_i + \sigma_i = \varrho_0 v_i \tag{5}$$

where ϱ_0 might be termed the "force of exploitation" $(1 + \sigma_i/v_i)$.[7] As far as the authors are aware, there is no substantive statement in Marxian literature that could serve to justify this postulate.[8] Indeed, there is no reason at all why the ratio of "unpaid" to "paid" labour should not be higher in some sectors (e.g., capital-intensive industries) than in others, even when both both components are expressed in terms of "value." It is clear, however, that *some* postulate of the type of (5) is necessary to make the value concept quantifiable and determinate. The choice of the particular form of (5) is therefore best regarded as a hidden part of the *definition* of "value," rather than an independent postulate that could be changed at will. No doubt a *uniformly* exploited proletariat fits better into the Marxian scheme of things than a society where differential rates of exploitation obscure the presumptively fundamental division between owners and non-owners of means of production.

[7] By analogy with the term "force of interest" which is sometimes used for one plus the rate of interest.

[8] There is, however, an interesting set of special assumptions (due to Professor N. Okishio) from which the postulate could be made to follow:

(a) Suppose the wage-bill paid out in each sector is proportional to the direct labour services $(v + \sigma)$ used in that sector (the wage differentials of capitalism being accepted as proper weights for various labour skills), i.e., $v + \sigma = \alpha v$.

(b) Suppose, further, that each worker spends his total wage (subsistence) on the same commodities in identical proportions, so that the "value" of the wage goods consumed in each sector (v) is proportional to the wage-bill paid out in that sector, i.e., $v = \beta v$. Then, by virtue of (a), equation (5) must hold, i.e., $\varrho_0 = \alpha/\beta$.

While these assumptions do not lack a certain plausibility at first sight, they would not in the opinion of the authors bring the Marxian model any closer to reality. Assumption (a) maintains that although capitalism "distorts" the valuation of *commodities*, it somehow succeeds in valuing *labour skills* correctly, while assumption (b) is much more restrictive than appears at first sight: It is perfectly possible to accept the subsistence theory of wages without holding that each worker must have the same consumption pattern (workers in the linen industry may "subsist" mainly on potatoes; those in the jute industry mainly on rice—two wage goods which differ greatly in labour content). Insofar as the two assumptions forfeit their claim to reality they must be regarded as an alternative statement, rather than a justification, of postulate (5).

The general transformation of price into value

Equation (5) allows us to put (4) in the form:

$$c'\omega + \varrho_0 v = \omega \,. \tag{6}$$

In this form the output values ω are already completely determinable from a knowledge of the price system. We need only assume that the workers' portion of each final output f_i can be allocated among the various sectors on the same lines as the producers' portions $c'w$. Let d_{ij} be the proportion of ω_i (or w_i) consumed out of wages by the workers of sector j. Then, by virtue of the definition of v_j as a subsistence wage (allowing no saving), we must have

$$v = d'\omega \,, \tag{7}$$

and equation (6) may be written as

$$(c' + \varrho_0 d')\,\omega = \omega \,. \tag{8}$$

It follows that ω can be obtained by solving $[(I - c') - \varrho_0 d']\,\omega = 0$, a system of equations whose consistency demands the vanishing of the determinant $|(I - c') - \varrho_0 d'|$. Pre-multiplying by $(I - c')^{-1}$ and dividing by ϱ_0,[9] the latter requirement may be put in the form:

$$[1/\varrho_0 I - \bar{c}'d']\,\omega = 0$$

where $\bar{c}' = (I - c')^{-1}$. Thus $1/\varrho_0$ is a characteristic root of $\bar{c}'d'$, and ω is its eigen-vector associated with $1/\varrho_0$. Suppose now c' is indecomposable. Then $\bar{c}'$ is positive because c' is an indecomposable matrix of the Leontief type. Since d' is nonnegative and nonzero, the matrix $\bar{c}'d'$ is also nonnegative and nonzero, so that there is a characteristic root of $\bar{c}'d'$ corresponding to a nonnegative eigen-vector; furthermore, this root is positive and largest in modulus of all the characteristic roots of $\bar{c}'d'$.[10] Thus , in order to satisfy the requirement of nonnegativity $(\omega \geqslant 0)$, $1/\varrho_0$ must be determined as the dominant positive root of the matrix $\bar{c}'d'$. Hence ϱ_0 is positive. Substituting this ϱ_0 in (8) and solving for ω,[11] we obtain the complete list of output-values, fully determined but for a proportionality factor which depends on whether labour-time is measured in hours, days, or any other units.

Corollaries of the general transformation

So far we have assumed that the pices involved in (1) were actual market

[9] It is clear that ϱ_0 cannot vanish, for that would imply $|I - c'| = 0$, i.e., the dominant root of c' would be unity, which would contradict the Leontief character of c'.

[10] See G. Debreu and I. N. Herstein, "Nonnegative Square Matrices," *Econometrica*, Vol. 21, 1953.

[11] We can show that the solution ω is strictly positive.

prices generating profit margins that vary from sector to sector, as experience teaches. Since, however, the transformation formula (8) has been derived without reference to the original processes of price formation, there is nothing to prevent us from applying the same formula to *any* price system, real or imaginary, based on arbitrary patterns of profit margins of our own choosing. One such pattern is of particular interest: the "prices of production" posited by Marx as the long-term gravity centres of capitalist market prices, are defined to entail *equal* profit margins[12] in all sectors. If we therefore assume system (1) to be couched in "production prices," the outputs w_j must be constant multiples of the total cost prices of material inputs and labour k_j $(= \sum w_{ij} + v_j)$, i.e.,

$$q_0(c' + d')w = w \tag{9a}$$

where q_0 is the constant "force of profit," which is the "rate of profit" p_i *plus* unity. The consistency of (9a) demands $|I - q_0(c' + d')| = 0$, and it follows at once that[13] $1/q_0$ is the dominant characteristic root of the matrix $c' + d'$. If we now recall that the reciprocal of the "force of exploitation" $(1/\varrho_0)$ has already revealed itself as the dominant characteristic root of the closely related matrix $\bar{c}'d'$, it will be clear at once that there must be some connection between "profit" (in production prices) and "exploitation" (in terms of value) which has not so far been studied.

To explore this we shall first define a unit-sum vector u consisting of positive elements, such that:

$$q_0 u'(c' + d') = u' \,. \tag{9b}$$

Since $1/q_0$ is the dominant characteristic root of $c' + d'$, a nonnegative matrix being indecomposable, it must be possible to find a positive solution for u' completely determined but for a proportionality factor. The latter can always be chosen so that $\sum u_i = 1$.

Premultiplying (8) by u' we find that:

$$\begin{aligned} u'\omega = u'(c' + \varrho_0 d')\omega &= u'[q_0(c' + d') + (1 - q_0)c' + (\varrho_0 - q_0)d']\omega \\ &= q_0 u'(c' + d')\omega + u'[(1 - q_0)c' + (\varrho_0 - q_0)d']\omega \,. \end{aligned}$$

Since, by virtue of (9b), the first term on the extreme right-hand side equals $u'\omega$, we must have:

$$u'[(1 - q_0)c' + (\varrho_0 - q_0)d']\omega = 0 \,,$$

[12] We prefer the familiar term "profit margin" to the somewhat misleading expression "rate of profit" which Marxists use for the ratio of profits to cost. We shall, however, use the latter (in the same sense) when historical theorems or preconceptions are at issue.

[13] Since the eigen-vector w is positive, $1/q_0$ must be the dominant (positive) root.

or by virtue of (7)

$$(\varrho_0 - q_0)\, u'v = (q_0 - 1)\, u'c'\omega \, .$$

Further, since $q_0 - 1$ is the rate of profit p_0, and $\varrho_0 - 1$ is the rate of exploitation ε_0, this can be put in the form

$$p_0 = \frac{u'v}{u'\varkappa}\,\varepsilon_0 \tag{9c}$$

where $\varkappa$ is the vector of cost of production $c'\omega + v$.

By virtue of the fact that u is positive and of unit sum, the fraction in (9c) represents the ratio of a weighted average of variable capitals (wage costs) to a similarly weighted average of total capitals (total costs), and is therefore obviously < 1. Accordingly equation (9c) allows us to draw two important conclusions: First, the rate of exploitation will always *exceed* the rate of profit; and secondly, a Marxian type of "economic progress" (where capital accumulation steadily reduces the share of wages in the total costs of all sectors) will normally entail a fall in the rate of profit, unless accompanied by a countervailing increase in the rate of exploitation.[14]

The formal correctness of these propositions does not, however, vindicate their political and ethical flavour, which is heavily dependent on the use of such emotive terms as "surplus" and "rate of exploitation." More particularly, if the subsistence theory of wages is abandoned, workers might save, and equation (7) may cease to be true. At the same time the proposition that economic progress must entail a falling share of wages in total costs loses much of its former plausibility.[15] Finally, a rising "rate of exploitation" may come to mean no more than universally growing savings propensities in the wake of increased prosperity.

[14] It should be noted that equation (9c) differs from Mrs. Robinson's interpretation of the Marxian profit-depressing mechanism (Joan Robinson, *An Essay in Marxian Economics*, p. 42). Her formula links the rate of exploitation with the rate of profit *in value terms*, say $\pi_0 = (v_0/\varkappa_0)\,\varepsilon_0$ which, as she acknowledges, is a mere tautology. But the *money* rate of profit p_0, as equation (9c) shows, is not strictly proportional to $v_0/\varkappa_0$, even if the rate of exploitation remains constant; for the *modus operandi* of "economic progress," even if it reduced $v/\varkappa$ by the same percentage in all sectors, could hardly fail to affect the matrix and therefore its latent vector u' (see (9b)), thus altering the weights attaching to the various $v/\varkappa$. The decline in p_0 (with constant ε_0) is therefore no more than a strong presumption.

[15] While progress may well bring falling wage-shares in the costs of most industries, particularly manufacturing, experience suggests that this is normally counteracted by the growing relative importance of the most wage-intensive sectors, e.g., retail trading, passenger transport, services, etc. It is by no means certain, therefore, in which direction the ratio $u'v/u'\varkappa$ in (9c) will move.

The transformation in a special model

The general solution we have outlined ((8) and (9)), though necessary to prove the determinacy of the "inverse transformation problem," is somewhat too complicated to serve as a basis for the aggregation problem which will be our main concern in the later sections of this article. It is therefore convenient to construct a simpler *particular* solution, by adding the following two assumptions:

$$v_i = v_i \tag{10}$$

and

$$\Sigma \phi_i = \Sigma f_i \, . \tag{11}$$

The first of these requires that the wage-bill paid out in any sector should be acceptable as a true measure of the (skill- and effort-weighted) labour power[16] expended there, i.e., that labour should not earn quasi-rents in any sector of the economy. This would seem to be quite consistent with the competitive labour markets of classical models. The quantity of labour involved is here expressed in terms of one-£-earning time periods.

The second assumption (11) is our interpretation of the Marxian dictum that "total price equals total value." Though stated in the relevant passage of *Das Kapital*, this postulate is peripheral to Marxian value theory. It is implicitly contradicted by other statements in Marx's work, many of which could equally well have been invoked in its stead. But its effect on the transformation is confined to the proportionality factor of the solution, and the algebraic convenience gained from it is certainly not bought at too damaging a price.

Since (11) implies $\Sigma(v+\sigma) = \Sigma(v+s)$, it follows from (5) and (10) that:

$$\varrho_0 = \frac{\Sigma\,(v+s)}{\Sigma\, v} \equiv r_0 \, . \tag{12}$$

We can therefore save ourselves the awkward calculation of ϱ_0 from the latent root of (8) and substitute for it the simple ratio r_0 which is directly ascertainable from the price system (1). By virtue of (10) and (12), equation (6) becomes

$$c'\omega + r_0 v = \omega \tag{13}$$

[16] It is essential to realize that Marxian "labour power" represents merely the *compensated* (or "paid for") labour as measured by the value of the consumer goods (i.e., wage) required to "regenerate" that labour power when it has been expended. The *total* value-contribution of labour $v_i + \sigma_i$ exceeds this sum by the surplus σ_i which is "expropriated" by the owners of capital.

which can be solved by simple matrix inversion:

$$\omega = r_0 \bar{c}' v \,. \tag{14}$$

In the special case treated by Cameron[17] it is assumed that the profits of all sectors vanish,[18] i.e., $s_i = 0$ and $r_0 = 1$. The solution for ω in (14) will then be identical with that for w implied in the original price system $c'w + v = w$, and therefore $\omega = w$.

2

The aggregation problem[19]

We shall now assume that the statistics of the price system available for our calculations do not reach down to the "primitive" sectors, but are based on groupings of the latter into "departments" as follows:

Sectors				*Department*
$1,$	$2,$	$\dots, a$	$\rightarrow$	A
$a+1,$	$a+2,$	$\dots, b$	$\rightarrow$	B
$\dots$	$\dots$	$\dots$		$\cdot$
$\dots$	$\dots$	$\dots$		$\cdot$
$m+1,$	$m+2,$	$\dots, n$	$\rightarrow$	N

Thus the output of department J appears as $W_J = w_{i+1} + w_{i+2} + \dots + w_j$.

A grouping of this sort is most conveniently defined by the "aggregation" matrix A where

$$A \equiv \begin{pmatrix} 1 \dots 1 & 0 \dots 0 \dots 0 \dots 0 \\ 0 \dots 0 & 1 \dots 1 \dots 0 \dots 0 \\ \dots\dots & \dots\dots\dots\dots \\ 0 \dots 0 & 0 \dots 0 \dots 1 \dots 1 \end{pmatrix} . \tag{15}$$

The "aggregator" (15) is an $N \times n$ matrix whose Jth row consists of i zeros

[17] B. Cameron, *op. cit.*

[18] $s_i = 0$ implies of course $\sigma_i = 0$, which produces a Marxian system without exploitation—almost a contradiction in terms.

[19] Previous discussions of aggregation in Leontief models include: M. Hatanaka, "Note on Consolidation within a Leontief System," *Econometrica*, Vol. 20, No. 2, 1952; J. B. Balderson and T. M. Whitin, "Aggregation in the Input-Output Model," *Economic Activity Analysis*, ed. by Morgenstern, 1954; E. Malinvaud, "Aggregation Problems in Input-Output Models," *The Structural Interdependence of the Economy*, ed. by Barna, 1954; M. McManus, "General Consistent Aggregation in Leontief Models," *Yorkshire Bulletin*, Vol. 8, No. 1, 1956; McManus, "On Hatanaka's Note on Consolidation," *Econometrica*, Vol. 24, No. 4, 1956; H. Theil, "Linear Aggregation in Input-Output Analysis," *Econometrica*, Vol. 25, No. 1, 1957; K. Ara, "The Aggregation Problem in Input-Output Analysis," *Econometrica*, Vol. 27, No. 2, 1959.

followed by $j-i$ units and $n-j$ zeros. Its function is shown by the obvious formulae:

$$(16) \qquad \begin{aligned} W &= Aw\,, & F &= Af\,, \\ \Omega &= A\omega\,, & V &= Av\,, \end{aligned}$$

where we follow the convention of denoting departmental aggregations by the capital letter X corresponding to the aggregated sector elements x.

To undo the work of the aggregation when occasion demands, we need the two "disaggregators" defined (in transposed form) as

$$(17) \qquad \begin{aligned} D' &\equiv \begin{pmatrix} \frac{w_1}{W_A} \cdots \frac{w_a}{W_B} & 0 \;\ldots\; 0 \;\ldots & 0 \;\ldots\; 0 \\ 0 \;\ldots\; 0 & \frac{w_{a+1}}{W_B} \cdots \frac{w_b}{W_B} \cdots & 0 \;\ldots\; 0 \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ 0 \;\ldots\; 0 & 0 \;\ldots\; 0 \;\ldots & \frac{w_{m+1}}{W_N} \cdots \frac{w_n}{W_N} \end{pmatrix} = \hat{W}^{-1} A \hat{w}\,, \text{ and} \\ \Delta' &\equiv \begin{pmatrix} \frac{\omega_1}{\Omega_A} \cdots \frac{\omega_a}{\Omega_A} & 0 \;\ldots\; 0 \;\ldots & 0 \;\ldots\; 0 \\ 0 \;\ldots\; 0 & \frac{\omega_{a+1}}{\Omega_B} \cdots \frac{\omega_b}{\Omega_B} \cdots & 0 \;\ldots\; 0 \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ 0 \;\ldots\; 0 & 0 \;\ldots\; 0 \;\ldots & \frac{\omega_{m+1}}{\Omega_N} \cdots \frac{\omega_n}{\Omega_N} \end{pmatrix} = \hat{\Omega}^{-1} A \hat{\omega}\,, \end{aligned}$$

where "capped" symbols stand for the diagonal matrices of the w_j, W_J, ω_j and Ω_J; the effect of D and Ω as multipliers are obvious:

$$(18) \qquad \begin{aligned} DW &= w\,, & AD &= I\,, \\ \Delta\Omega &= \omega\,, & A\Delta &= I\,. \end{aligned}$$

Finally, the sectoral output quotas (3) may be consolidated into interdepartmental quotas (in price- or value-terms) as follows:

$$(19) \qquad \begin{aligned} C &= D'cA'\,, \\ \Gamma &= \Delta'\gamma A' = \Delta' c A'\,. \end{aligned}$$

This completes our armoury of mathematical tools, and we can now tackle the substantive problem at issue. Let us begin by stating it more precisely.

By the "true translation of W" we shall mean Ω, i.e., the result of *first* transforming w into value-terms ω (13), and *afterwards* aggregating ω into Ω. By the "rough translation of W" we shall mean $\tilde{\Omega}$, i.e., the result of *first* aggregating the price system ((16) and (19)), and *afterwards* transforming into values by applying the method (13) to the aggregated structure, i.e.,

$$C'\tilde{\Omega} + r_0 V = \tilde{\Omega}\,.\text{[20]} \tag{20}$$

In order to compare the "rough translation" with the "true" one, we must find some equation in Ω which can be compared with (20). This can be done by premultiplying (13) by A, and writing (in view of 16): $Ac'\omega + r_0 V = \Omega$. Substituting for ω from (18) this becomes:

$$(\Delta' cA')'\Omega + r_0 V = \Omega = \Gamma'\Omega + r_0 V\,. \tag{21}$$

It is clear from (20) and (21) that the "rough" and "true" translations of the departments' outputs ($\tilde{\Omega}$ and Ω) will not in general coincide. It can, however, be shown that the totals for the *national income* resulting from the two translations will always be the same: the net incomes of the departments, i.e., the difference between their outputs and their material inputs ("constant capitals"), are given by $\Omega - \Gamma'\Omega$ in the "true," and $\tilde{\Omega} - C'\tilde{\Omega}$ in the "rough" translation, and it is clear from (20) and (21) that these are equal to $r_0 V$ in both cases. It follows that the sum-total of these elements (which is the net product $\Sigma\, \Phi_J$) must also be the same in "rough" and "true" translation, i.e.,

$$\Sigma\, \tilde{\Phi}_J = \Sigma\, \Phi_J\,. \tag{22}$$

This is indeed no more than might have been expected. Since the "value" of the national income is by definition equal to the sum-total of labour expended it must remain unaffected by the way in which that labour is allocated among departments, and proposition (22) becomes self-evident.

However, all the *other* elements of the interdepartmental value flow, along with their sums and sub-totals, will in general turn out to be different according to whether the "rough" or the "true" translation is used.

"Non-distorting" aggregation

A comparison of (20) and (21) may be expected to yield some information on the "distortions of aggregation" $\tilde{\Omega} - \Omega$. Where the aggregation is such that "rough" and "true" translations coincide ($\tilde{\Omega} = \Omega$) we shall speak of "non-distorting aggregation." It is clear from (20) and (21) that a necessary and sufficient condition for such a situation may be stated as

$$C'\tilde{\Omega} = (\Delta' cA')'\Omega\,.$$

This immediately yields a simpler sufficient (though not necessary) condition:

$$C = D'cA' = \Delta' cA' = \Gamma\,. \tag{23}$$

[20] It is clear that r_0 is unaffected by the aggregation since $\Sigma\,(v+s)/\Sigma\, v = \Sigma\,(V+S)/\Sigma\, V$.

In words: *An aggregation pattern of sectors into departments will be non-distorting if the resulting interdepartmental output quotas are the same in value terms as they are in price terms.*

In this general form the criterion is of little value as it presupposes a detailed knowledge of the original value system, which is precisely what the criterion should help us to dispense with. We can, however, improve matters by deriving two rather more stringent conditions from the general form: First, it is clear that (23) will be fulfilled, if

$$D = \Delta \, .$$

In words: *The aggregation will be non-distorting if the relative importance of the constituent sectors within each department is the same in value-terms as it is in price-terms.* The condition is of course more stringent than is necessary or desirable, but the knowledge of the original value system which it presupposes is only, as it were, one-dimensional, and might on occasion be supplied by intuition.

Secondly, we can derive a condition for non-distorting aggregation that is independent of any knowledge of the value system at all:

(24) $$C'A = Ac' \, .$$

As post-multiplication by Δ shows, (24) implies $C'A\Delta = Ac'\Delta$. Since $A\Delta = I$, it follows that $C' = \Gamma'$, i.e., that the general condition (23) must be fulfilled. Condition (24) might be expressed as follows: *Aggregation is non-distorting if only those sectors are aggregated into departments whose (departmental) output distributions follow the same pattern.*

A further sufficient condition, this time affecting *input* coefficients, may be derived by analogy with (24), and stated as

(25) $$\begin{aligned} DC' &= c'D \qquad \text{and} \\ DV &= v \, . \end{aligned}$$

Its meaning becomes clear when we substitute for D from (17) and reformulate it as

(26a) $$A'K' = k'A' \qquad \text{and}$$
(26b) $$A'L = l$$

where K and k stand for the input coefficients ($\hat{W}C\hat{W}^{-1}$ and $\hat{w}c\hat{w}^{-1}$) and L and l for the corresponding labour-input ratios ($\hat{W}^{-1}V$ and $\hat{w}^{-1}v$). Condition (25) is therefore equivalent to the postulate that *only those sectors be aggregated into departments whose (departmental) cost structures, including wages, follow the same pattern.*

To show that (25) is in fact a sufficient condition for non-distorting aggregation, we premultiply (20) by D obtaining in view of (25):

$$DC'\tilde{\Omega} + r_0 DV = c'(D\tilde{\Omega}) + r_0 v = D\tilde{\Omega} .$$

But this is identical with the equation which must be true of ω (see (13)), and we can therefore write:

$$D\tilde{\Omega} = \omega = \Delta\Omega \qquad \text{(see (18))} .$$

Finally, premultiplying by A, and remembering that by definition $AD = A\Delta = I$, we have $\tilde{\Omega} = \Omega$, i.e., an aggregation pattern obeying condition (25) will always be non-distorting.

Our predecessors in the study of the aggregation problem in Leontief models have shown that if (26a) is satisfied, output predictions derived by means of the aggregate input-output table contain no forecasting errors that are to be ascribed to the aggregation.[21] It may be of some interest to remark that, on the understanding that (26b) is also satisfied, an aggregation designed to give true output predictions for all final-bill vectors transforms W into value-terms Ω without any aggregation bias.

Aggregation with limited distortions

The foregoing analysis invites a form of generalization in which the equalities obtained as conditions for non-distorting aggregation are "relaxed" into inequalities specifying *ranges* rather than definite par values for the critical coefficients. We can then attempt to derive from these ranges certain consequential limits within which the distortions $\tilde{\Omega} - \Omega$ must lie.

In the interests of conciseness we shall henceforth denote the Leontief inverse of a matrix x by placing a bar on it, i.e., $\bar{x} \equiv (I - x)^{-1}$ and $\bar{x}' \equiv (I - x')^{-1}$. Premultiplying (14) by A, we obtain the "true translation":

$$\Omega = r_0 A \bar{c}' v . \tag{27}$$

Similarly, the "rough translation" as defined in (20) can be written as

$$\tilde{\Omega} = r_0 \bar{C}' V . \tag{28}$$

We can therefore express the distortion as the difference between (28) and (27), i.e.,

$$\tilde{\Omega} - \Omega = r(\bar{C}'V - A\bar{c}'v) . \tag{29}$$

It is this expression whose value we wish to place within bounds.

We shall now assume that although the (departmental) output quotas of the microsectors ($c_{tJ} \equiv \sum_{t \varepsilon J} c_{tt}$) are not known, it is possible to assign lower and upper bounds to them within each separate department I, i.e.,:

[21] See references in footnote 19.

$$C^{*}_{IJ} \leqslant \min_{r\varepsilon I} (c_{rJ}) = \min_{r\varepsilon I} (\sum_{t\varepsilon J} c_{rt}) ,$$

$$C^{**}_{IJ} \geqslant \max_{r\varepsilon I} (c_{rJ}) = \max_{r\varepsilon I} (\sum_{t\varepsilon J} c_{rt}) .$$

It is clear that both the *unknown* micro-quotas c_{rJ} and the *known* macro-quotas C_{IJ} (which are averages of the former) must be "bounded" by the extreme values defined above, i.e.,

(30a) $$C'_{*}A \leqslant Ac' \leqslant C'_{**}A$$

and

(30b) $$C'_{*} \leqslant C' \leqslant C'_{**} .$$

Since C' is a Leontief-type matrix and has dominant root < 1, the same will *a fortiori* hold of C'_{*} (which is smaller), but not necessarily of C'_{**}. In what follows, however, *we shall assume that C'_{**} is also of the Leontief-type* (*dominant root* < 1). Armed with this assumption we can prove from (30a) that[22]

(31a) $$\bar{C}'_{*}A \leqslant A\bar{c}' \leqslant \bar{C}'_{**}A .$$

Similarly, it follows from (30b) that

(31b) $$\bar{C}'_{*} \leqslant \bar{C}' \leqslant \bar{C}'_{**} .$$

Postmultiplying (31a) by v and (31b) by V, we have by virtue of (16):

(32a) $$\bar{C}'_{*}V \leqslant A\bar{c}'v \leqslant \bar{C}'_{**}V$$

and

(32b) $$\bar{C}'_{*}V \leqslant \bar{C}'V \leqslant \bar{C}'_{**}V .$$

Since the middle terms of (32a) and (32b) are both bounded by the same limits, their difference must be smaller than the gap between these limits, i.e.,[23]

(33) $$r_0\,|\bar{C}'V - A\bar{c}'v| = |\tilde{\Omega} - \Omega| \leqslant r_0(\bar{C}'_{**} - \bar{C}'_{*})\,V .$$

[22] Since by virtue of (30a):

$$Ac'^{s} = (Ac')\,c'^{s-1} \geqslant C'_{*}Ac'^{s-1} \geqslant C'^{2}_{*}Ac'^{s-2} \geqslant \dots \geqslant C'^{s}_{*}A$$

and

$$Ac'^{s} = (Ac')c'^{s-1} \leqslant C'_{**}Ac'^{s-1} \leqslant C'^{2}_{**}Ac'^{s-2} \leqslant \dots \leqslant C'^{s}_{**}A ,$$

the formula (31a) follows directly from the assumed convergence of the series

$$\bar{c}'(= I + c' + c^{2} + \dots),\ \bar{C}'_{*} \quad \text{and} \quad \bar{C}'_{**} .$$

[23] Throughout the rest of this article the vertical bars will denote absolute values, and not determinants.

Thus, if the departmental output quotas of aggregated micro-sectors, though not individually known, can be assigned appropriate ranges, the distortion of the aggregation will not exceed certain definite limits which can be calculated from these ranges. The theorem does, however, depend on the assumption that C'_{**} has dominant root < 1. When the output quotas of the aggregated micro-sectors are all equal, the model merges into that discussed in (24), and the distortion vanishes.

A similar result *mutatis mutandis* may be formulated in terms of the input coefficients k_{ij} and l_j. Let

$$K^*_{IJ} \leqslant \min_{t\varepsilon J} (k_{It}) = \min_{t\varepsilon J} (\sum_{r\varepsilon I} k_{rt}) \; ; \qquad L^*_J \leqslant \min_{t\varepsilon J} (l_t) \; ;$$

$$K^{**}_{IJ} \geqslant \max_{t\varepsilon J} (k_{It}) = \max_{t\varepsilon J} (\sum_{r\varepsilon I} k_{rt}) \; ; \qquad L^{**}_J \geqslant \max_{t\varepsilon J} (l_t) \; .$$

Assume that K_{**} is of the Leontief type (dominant root < 1). Then we have the formula:

$$|\tilde{\Omega} - \Omega| = r_0 \, |\hat{W}\bar{K}'L - A\hat{w}\bar{k}'l| \leqslant r_0 \hat{W} (\bar{K}'_{**} L_{**} - \bar{K}'_{*} L_{*}) \; .$$

The distortion is therefore "bounded" by the ranges assigned to the unknown sectoral input coefficients within each department.[24]

3

The special case of Marx's departments

The foregoing conclusions can easily be applied to the extreme form of aggregation implied by Marx in which only two departments A and B

[24] A similar problem of much wider applicability arises when we wish to forecast the total outputs of an economy in some future year (say W_1) from the final demands expected in that year (F_1), i.e., $KW_1 + F_1 = W_1$; hence $W_1 = \bar{K}F_1$. It is assumed that our knowledge is restricted to the macro-system of some base period 0 ($K_0 = AkD_0$). It is clear that the *future* macro-system $K_1 (K_1 = AkD_1)$ though built up from unaltered micro-coefficients k, may differ from K_0 because of changes in the intra-departmental output structure ($D_0 \rightarrow D_1$). Our forecast of the future outputs can therefore only be an estimate: $\tilde{W}_1 = \bar{K}_0 F_1$, while the "true" Leontief forecast should be $W_1 = \bar{K}_1 F_1$.

Required: the limits of the distortion:

$$|\tilde{W}_1 - W_1| = |(\bar{K}_0 - \bar{K}_1) F_1| = |(\bar{K}_0 A - A\bar{k}) f_1| \; .$$

We define, as in the text, minima K_* and maxima K_{**}. Then we obtain:

$$\bar{K}_* A f_1 \leqslant A\bar{k} f_1 \leqslant \bar{K}_{**} A f_1 \, ,$$
$$\bar{K}_* A f_1 \leqslant \bar{K}_0 A f_1 \leqslant \bar{K}_{**} A f_1 \; .$$

Substraction by analogy with (33) shows that the distortion will be bounded as follows:

$$|\tilde{W}_1 - W_1| \leqslant (\bar{K}_{**} - \bar{K}_*) \, F_1 \; .$$

Again the assumption that K_{**} is of the Leontief type (dominant root < 1) is crucial.

survive ("mens of production" and "means of consumption"), and in which the output of any B-sector is by definition wholly absorbed in final consumption.

In this particular case the aggregated value-scheme (2) can be written as

$$(34)\qquad \begin{aligned} \Omega_{AA} + \Omega_{AB} + \Phi_A &= \Omega_A \\ \Phi_B &= \Omega_B \\ V_A + V_B &= V_0 \\ \Sigma_A + \Sigma_B &= \Sigma_0 \end{aligned}$$

where Ω_{AA} and Ω_{AB} are the aggregated amounts of constant capital ω_{ij}; V_A and V_B the aggregated amounts of variable capital $v_j(= v_j)$; and Σ_A and Σ_B are the aggregated surpluses σ_j. It is clear from the definition that the net products Φ_A and Φ_B will be identical with investment and consumption, respectively.

To find the nexus linking the "rough" and "true" translations of the above quantities we combine equations (20) and (21) into

$$r_0 V = (I - C')\tilde{\Omega} = (I - \Gamma')\Omega$$

which can be written *in extenso* as

$$\left[\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}C_{AA} & 0\\ C_{AB} & 0\end{pmatrix}\right]\begin{Bmatrix}\tilde{\Omega}_A\\ \tilde{\Omega}_B\end{Bmatrix} = \left[\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}\Gamma_{AA} & 0\\ \Gamma_{AB} & 0\end{pmatrix}\right]\begin{Bmatrix}\Omega_A\\ \Omega_B\end{Bmatrix}.$$

The first of these equations is

$$(35)\quad (1 - C_{AA})\tilde{\Omega}_A = (1 - \Gamma_{AA})\Omega_A \qquad \text{or} \qquad \tilde{\Omega}_A - \tilde{\Omega}_{AA} = \Omega_A - \Omega_{AA}.$$

From the last version it follows evidently that

$$(36)\qquad \tilde{\Omega}_{AB} + \tilde{\Phi}_A = \Omega_{AB} + \Phi_A$$

i.e., the "rough" translation of the net output of department A must coincide with its "true" translation. In other words, the net output of A will always be "correctly estimated."

From (35) it follows that

$$\frac{\tilde{\Omega}_A}{\Omega_A} = \frac{1 - \Gamma_{AA}}{1 - C_{AA}} \qquad \text{and} \qquad \frac{\tilde{\Omega}_{AA}}{\Omega_{AA}} = \frac{C_{AA}}{1 - C_{AA}} \div \frac{\Gamma_{AA}}{1 - \Gamma_{AA}}.$$

Accordingly, *both output-value and constant capital in department A will be over- (or under-) estimated if the money ratio of constant capital to total output in A over- (or under-) states the equivalent ratio in value terms*. Moreover, it is clear from the second version of (35) that any distortion of the constant capital Ω_{AA} say α, will entail a distortion of exactly the same magnitude and direction in the total output Ω_A.

Further, we know from (22) that the national income $\Phi_A + \Phi_B$ will always be estimated correctly. It follows that any positive distortion in consumption, say β, must be exactly counterbalanced by a negative distortion in investment (Φ_A).

In view of (36) the only possible pattern of distortion in the first two lines of (34) can therefore be presented as

$$\begin{array}{cccl} \alpha & \beta & -\beta & = \alpha\,, \\ & & \beta & = \beta\,, \end{array} \tag{37}$$

where α and β may be positive or negative, and of equal or unequal sign.

In the particular case of "simple reproduction," i.e., zero investment, we must have $\Phi_A = 0$, and the distortion β vanishes. It is clear from (37), therefore, that total consumption Φ_B and the constant capital of B will be correctly estimated, and that only the output value of A and its constant capital remain subject to distortion.

Expansion in scale

Further inferences concerning the distortions α and β can be drawn if it is assumed that the volume and composition of investment (Φ_A) is such as to supply each sector with a strictly proportionate increase in *all* its inputs. In that case the final output of each A-sector f_i (= a component of Φ_A in price terms) can be split into n consignments f_{ij} each destined as investment input for a different sector j, and each proportionate to the current input w_{ij}. Thus:

$$f_{ij} = g_j w_{ij} = g_j c_{ij} w_i \tag{38}$$

where g_j is the uniform scale at which all inputs of sector j are planned to expand. It follows that the total investment input of sector j, say h_j, will be

$$h_j = \sum_{i=1}^{a} f_{ij} = g_j \sum_{i=1}^{a} c_{ij} w_i\,,$$

or in matrix form:

$$\begin{aligned} h &= gc'w \qquad &&\text{(in price terms)},\\ \psi &= gc'\omega \qquad &&\text{(in value terms)}, \end{aligned}$$

where g is the diagonal matrix of input-growth rates g_j. The aggregation of the sectoral investment inputs ψ will yield the true translation of the departmental investment inputs

$$\Psi = Agc'\omega = (Ag\Delta)\Omega - r_0 Agv \qquad \text{(see (13) and (18))}\,. \tag{39}$$

If the detailed sectoral system is unknown, we can only compute the "rough" translation from the two-by-two table of departments:

$$\tilde{\Psi} = G\tilde{\Omega} - r_0 GV \tag{40}$$

where G is the aggregated diagonal matrix of input-growth rates with elements F_{AA}/W_{AA} and F_{AB}/W_{AB}. The difference between (40) and (39) yields the distortion of investment inputs.

A case of particular interest arises when A-sectors and B-sectors have the same input-growth rates *inter se* (g_A and g_B), a postulate which can be put as:

$$Ag = GA \,. \tag{41}$$

Then, by virtue of the fact that $A\Delta = I$ (18), and $Av = V$ (16), equation (39) becomes

$$\Psi = G\Omega - r_0 GV \,. \tag{42}$$

The distortion of investment-inputs yielded by the subtraction of (42) from (40) is therefore

$$\tilde{\Psi} - \Psi = G(\tilde{\Omega} - \Omega) = \begin{Bmatrix} \alpha g_A \\ \beta g_B \end{Bmatrix} \tag{43}$$

where α and β are the distortions of the two outputs Ω_A and Ω_B (see (37)).

It is clear, however, that the sum of the investment inputs Ψ (or $\tilde{\Phi}$) must yield the total investment output Φ_A (or $\tilde{\Psi}_A$), and the distortion of the latter, $-\beta$ (see (37)), can therefore be computed from (43):

$$-\beta = \alpha g_A + \beta g_B \,.$$

It follows at once that

$$\frac{\beta}{\alpha} = -\frac{g_A}{1 + g_B} \,,$$

and three important inferences may be drawn for the case of uniform expansion (as defined by 38 and 41):

(1) The distortion in the output of departments A and B will always be in the opposite direction, i.e., if Ω_A is over-estimated, Ω_B will be under-estimated, and conversely.

(2) The distortion of the consumer output Ω_B will in all normal cases of growth (where $g_A < 1$) be smaller than the (opposite) distortion of the capital output Ω_A. It will vanish altogether when there is no growth in department A, i.e., $g_A = 0$.

(3) When both departments' constant capitals expand at the same rate ($g_A = g_B = g_0$), the relative distortion of Ω_B will be the smaller, the lower the uniform growth rate, i.e., $|\beta/\alpha| = 1/(1 + 1/g_0)$. It will be smaller than half in all normal cases of balanced growth (where $g_0 < 1$).

Osaka University and Nuffield College, Oxford