G. C. Harcourt and Vincent G. Massaro
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# NOTES AND MEMORANDA

## A NOTE ON MR. SRAFFA'S SUB-SYSTEMS[1]

1. IF the whole of the national income is absorbed by wage payments, " the relative values of commodities [in the economic systems of Mr. Sraffa's *Production of Commodities by Means of Commodities*] are in proportion to their labour cost, that is to say to the quantity of labour which directly and indirectly has gone to produce them." [2] To establish this proposition Sraffa refers the reader to Appendix A, " On Sub-Systems." We propose in this note to show, both diagrammatically and algebraically, how these sub-systems may be derived from the main economic system, in order to describe their general properties and, in particular, to illustrate the proposition quoted above. We proceed by defining the simple economic system of Sraffa's book and the sub-systems implied in it. We then derive the latter from the former. Finally, we establish the above proposition and mention some possible implications of the analysis.

2. The economic system contains commodities which enter into the production of both themselves and other commodities. Such commodities are called basic commodities, and they are distinguished from non-basics, commodities which do not enter into the production of other commodities. The system consists of industries which produce one commodity. The production cycle is one year. The gross product (or total output) of any industry in a self-replacing economic system may be either greater than or equal to the total amount of the commodity which is used to produce both itself and other commodities. Such an economic system is in a " self-replacing " state in the sense that it is *capable of*, but need not in fact be, replacing itself.[3] Commodities viewed from their aspect as inputs are called means of production. The excess of the total output of any commodity over that part of it which is equal to its use as a means of production is a component of the net product of the system. The sum of these components, when valued, is equal to the national income of the economy, which, in turn, is equal to the sum of the values of the total outputs less the value of the means of production. It can now be seen that an economy which has a positive or zero national income and in which the production of any basic commodity

[1] The writers are grateful to Mr. Sraffa for his helpful comments on a draft of this note.

[2] Piero Sraffa, *Production of Commodities by Means of Commodities* (Cambridge University Press, 1960), p. 12.

[3] While the economic systems of this paper are in fact in " self-replacing " states, it should be stressed that self-replacement is a property of the basic equations of the economic systems concerned and not necessarily of the economic systems themselves. This implies that there is at least one set of proportions in which the equations can be combined such that they are in a " self-replacing " state. (See Sraffa, *op. cit.*, p. 5, n. 2.)

is at least equal to the amount of it used as a means of production, is in a "self-replacing" state.

3. Each industry employs labour which is paid a uniform wage at the end of the period. The relative prices of commodities are such that, given the wage-rate and the technical conditions of production, each industry earns the same rate of profits on its means of production.[1] The rate of profits is the ratio of the excess of the value of the total output of the economy over the value of the wages and the means of production to the value of the means of production. The units in which values are measured need not concern us here, as most of the results below can be derived in terms of physical amounts of the same commodity (or, sometimes, of labour). However, it can be taken that there is a standard commodity which is independent of the rate of profits and the wage-rate, which can serve as the unit of measurement for the system as a whole.[2] Finally, the inputs of commodities and labour per unit of output of commodities are constants (but this must not be taken to imply that an assumption of constant returns to scale is made).

4. It must be stressed that the relationships in the economic system described here relate to one year only. They occur within the bounds of that year, and there is no necessary connection between them and the relationships that exist in other years, either past or present. In particular, it is *not* implied that the means of production come from the immediately preceding year; nor that those parts of the gross product which are *equal to* the means of production will be used as means of production in following years. The relationships apply to a particular period of time *and to no other.*

5. We shall consider a three-industry system producing three commodities, $a$, $b$ and $c$, in amounts such that each commodity is a component of the net product. This assumption is dropped in paragraph 11, and it is shown that it does not affect the propositions derived. We write the system as follows:

$$(1 + r)\ (x_{aa}Ap_a + x_{ab}Ap_b + x_{ac}Ap_c) + l_aAw \equiv Ap_a$$
$$(1 + r)\ (x_{ba}Bp_a + x_{bb}Bp_b + x_{bc}Bp_c) + l_bBw \equiv Bp_b$$
$$(1 + r)\ (x_{ca}Cp_a + x_{cb}Cp_b + x_{cc}Cp_c) + l_cCw \equiv Cp_c$$

where $r$ = rate of profits;
$w$ = wage-rate;
$p_i$ = price of commodity $i$, $(i = a, b, c)$;
$x_{ij}$ = input of commodity $j$ per unit of output of $i$, (both $i$ and $j = a, b, c$);
$l_i$ = input of labour per unit of output of $i$, $(i = a, b, c)$;

and $A$, $B$, $C$ are the gross products of $a$, $b$, $c$ respectively.

6. The physical conditions of production are shown in Fig. 1. "The

[1] The relative prices and the rate of profits are the solutions of a set of simultaneous equations.
[2] Sraffa, *op. cit.*, Chapters IV and V.

commodities forming the gross product . . . can be unambiguously distinguished as those which go to replace the means of production and those which together form the net product of the system."[1] This is shown by the shaded areas (the components of the net product of the system) and the clear

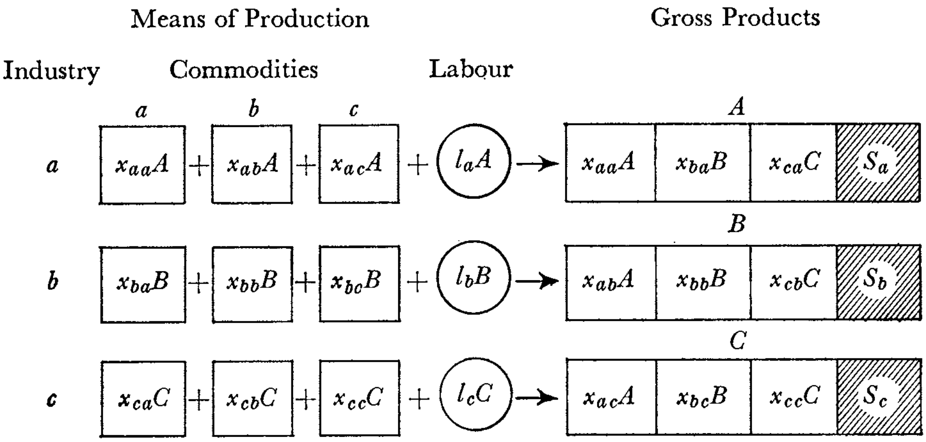


FIG. 1

*Note:* No significance should be attached to the sizes of each box or circle.

areas (the means of production) of the Gross Products column of Fig. 1. The components of the net product in physical terms are

$$S_a \equiv A - \alpha$$
$$S_b \equiv B - \beta$$
$$S_c \equiv C - \gamma$$

where

$$\alpha \equiv x_{aa}A + x_{ba}B + x_{ca}C;$$
$$\beta \equiv x_{ab}A + x_{bb}B + x_{cb}C;$$
$$\gamma \equiv x_{ac}A + x_{bc}B + x_{cc}C.$$

7. We now obtain the sub-systems from the main system. The main system can be divided into as many parts as there are commodities which are components of the *net* product, in such a way that each part is in a self-replacing state with a net product of one commodity only. Each part is called a sub-system—in our example there are three. The net product of each sub-system is equal to the amount of that commodity in the net product of the main system. The total amount of each commodity used as a means of production in the *three* sub-systems is equal to their use as means of production in the main system. Similarly, the same total amount of labour is used in the three sub-systems as in the main system; moreover, the labour used in (say) the three *a* producing industries of the sub-systems equals the

[1] Sraffa, *op. cit.*, p. 89.

amount of labour used in the $a$ industry of the main system. That is to say, the three sub-systems taken together are merely a rearrangement of the original system. The formation of the three sub-systems is illustrated in Fig. 2.

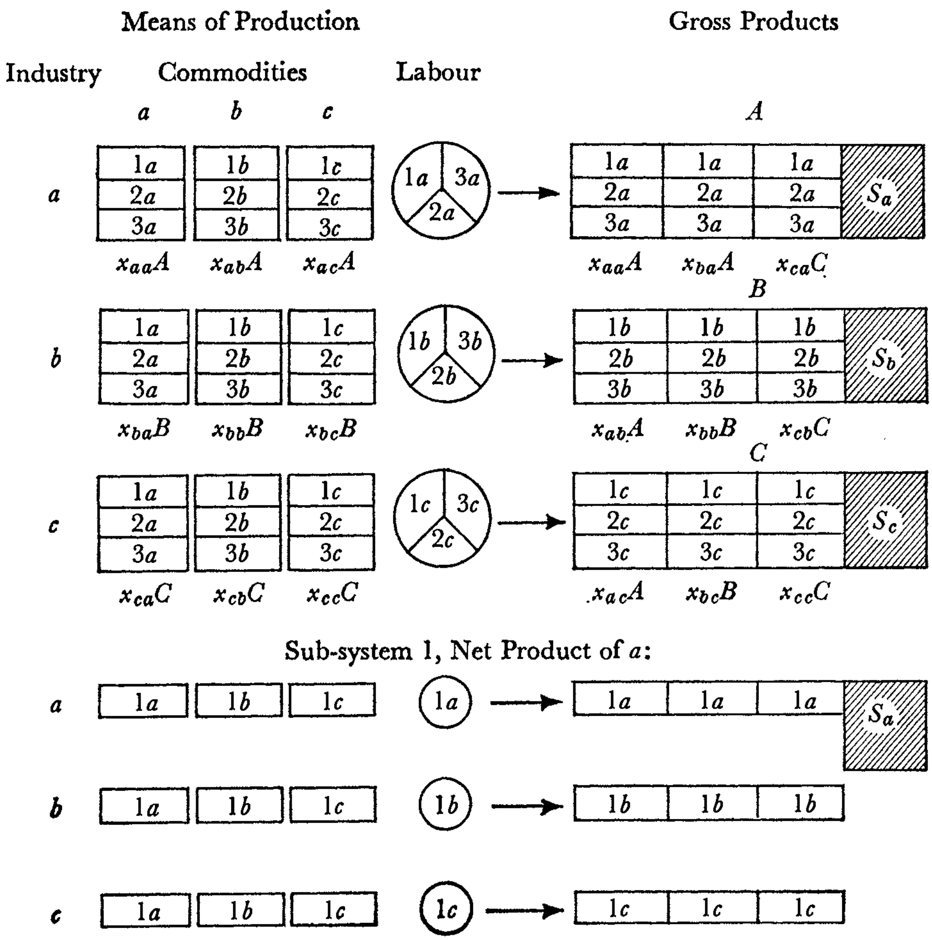


FIG. 2

The other two sub-systems can be built up in an analogous manner.

8. Algebraically, the sub-systems can be written as follows:

Sub-system 1

$$(1+r)\,(x_{aa}A'p_a + x_{ab}A'p_b + x_{ac}A'p_c) + l_aA'w \equiv A'p_a$$
$$(1+r)\,(x_{ba}B'p_a + x_{bb}B'p_b + x_{bc}B'p_c) + l_bB'w \equiv B'p_b$$
$$(1+r)\,(x_{ca}C'p_a + x_{cb}C'p_b + x_{cc}C'p_c) + l_cC'w \equiv C'p_c$$

where
$$x_{aa}A' + x_{ba}B' + x_{ca}C' \equiv \alpha';$$
$$x_{ab}A' + x_{bb}B' + x_{cb}C' \equiv \beta';$$
$$x_{ac}A' + x_{bc}B' + x_{cc}C' \equiv \gamma'.$$

$A'$, $B'$ and $C'$ are the gross products of the sub-system, which has a net product of $a$ equal to $S_a$; $\alpha'$, $\beta'$ and $\gamma'$ are the means of production of the sub-system; $(S_a + \alpha') = A' > \alpha'$; $B' = \beta'$ and $C' = \gamma'$. Sub-systems 2 and 3 can be similarly written, using $A''$, $B''$ and $C''$ and $\alpha''$, $\beta''$ and $\gamma''$ for the gross products and the means of production, respectively, of sub-system 2, $A'''$, $B'''$ and $C'''$ and $\alpha'''$, $\beta'''$ and $\gamma'''$ for the corresponding amounts for sub-system 3. There are twenty-seven unknowns in the three sub-systems: the nine gross outputs of the three sub-systems, the nine means of production and the nine labour inputs. That there are also twenty-seven independent equations can be illustrated by setting out the nine equations for sub-system 1, namely, the three gross output equations, the three means of production equations and the three labour equations.

*Gross Outputs*

$$A' = S_a + \alpha' \qquad \ldots \qquad (1)$$

$$B' = \beta' \qquad \ldots \qquad (2)$$

$$C' = \gamma' \qquad \ldots \qquad (3)$$

*Means of Production:*

$$\alpha' = \frac{x_{aa}S_a + x_{ba}\beta' + x_{ca}\gamma'}{1 - x_{aa}} \qquad \ldots \qquad (4)$$

$$\beta' = \frac{x_{ab}(S_a + \alpha') + x_{cb}\gamma'}{1 - x_{bb}} \qquad \ldots \qquad (5)$$

$$\gamma' = \frac{x_{ac}(S_a + \alpha') + x_{bc}\beta'}{1 - x_{cc}} \qquad \ldots \qquad (6)$$

These can be solved to give:[1]

$$\alpha' = \frac{S_a}{\Delta}\{x_{aa}[(1 - x_{bb})(1 - x_{cc}) - x_{cb}x_{bc}] + x_{ba}[x_{ab}(1 - x_{cc}) + x_{cb}x_{ac}] + x_{ca}[x_{ab}x_{bc} + x_{ac}(1 - x_{bb})]\}$$

$$\beta' = \frac{S_a}{\Delta}\{x_{ab}(1 - x_{cc}) + x_{cb}x_{ac}\}$$

$$\gamma' = \frac{S_a}{\Delta}\{x_{ac}(1 - x_{bb}) + x_{ab}x_{bc}\}$$

where $\Delta = (1 - x_{aa})[(1 - x_{bb})(1 - x_{cc}) - x_{cb}x_{bc}] + x_{ba}[- x_{ab}(1 - x_{cc}) - x_{cb}x_{ac}] - x_{ca}[x_{ab}x_{bc} + x_{ac}(1 - x_{bb})]$

and $\neq 0$.

The condition for $\Delta \neq 0$ is that the inputs of $a$, $b$ or $c$ into themselves should not be such as to absorb the whole gross outputs of themselves, *i.e.*, $\Delta \neq 0$ provided that $x_{aa}$ or $x_{bb}$ or $x_{cc} \neq 1$.

[1] We are indebted to Mrs. J. D. Frost for obtaining these solutions for us.

*Labour*

$$L_a' = l_a A' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$
$$L_b' = l_b B' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$
$$L_c' = l_c C'_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $L_a'$, $L_b'$ and $L_c'$ are the amounts of labour in the respective industries of the sub-system.

9. We now show that the national incomes of both the main system and the sub-systems are equal to wages plus profits. This can be seen by deducting the means of production of each industry from the left-hand and right-hand sides of the appropriate equations. In the main system we are left with:

$$r(x_{aa}Ap_a + x_{ab}Ap_b + x_{ac}Ap_c) + l_aAw \equiv Ap_a - (x_{aa}Ap_a + x_{ab}Ap_b + x_{ac}Ap_c)$$
$$r(x_{ba}Bp_a + x_{bb}Bp_b + x_{bc}Bp_c) + l_bBw \equiv Bp_b - (x_{ba}Bp_a + x_{bb}Bp_b + x_{bc}Bp_c)$$
$$r(x_{ca}Cp_a + x_{cb}Cp_b + x_{cc}Cp_c) + l_cCw \equiv Cp_c - (x_{ca}Cp_a + x_{cb}Cp_b + x_{cc}Cp_c)$$

and in sub-system 1,

$$r(x_{aa}A'p_a + x_{ab}A'p_b + x_{ac}A'p_c) + l_aA'w \equiv A'p_a - (x_{aa}A'p_a + x_{ab}A'p_b + x_{ac}A'p_c)$$
$$r(x_{ba}B'p_a + x_{bb}B'p_b + x_{bc}B'p_c) + l_bB'w \equiv B'p_b - (x_{ba}B'p_a + x_{bb}B'p_b + x_{bc}B'p_c)$$
$$r(x_{ca}C'p_a + x_{cb}C'p_b + x_{cc}C'p_c) + l_cC'w \equiv C'p_c - (x_{ca}C'p_a + x_{cb}C'p_b + x_{cc}C'p_c)$$

When $r = 0$ the whole of the national income is accounted for by wages, that is to say, the values added of the industries in both the main system and the sub-systems are equal to their respective wages bills. If we rearrange the commodities in the sub-systems so that from each gross product is subtracted those parts which equal the means of production we are left with net products of one commodity only, in sub-system 1, commodity $a$. Thus,

$$l_aA'w + l_bB'w + l_cC'w \equiv (A'p_a - \alpha'p_a) + (B'p_b - \beta'p_b) + (C'p_c - \gamma'p_c) \equiv S_ap_a$$

These net products, however, are each equal in value to the wages bills of the three industries in their respective sub-systems, that is, to the national incomes of each. In sub-system 1 *the net product is equal to the value of the labour which, directly in industry* a *and indirectly in industries* b *and* c, *produced it.* Moreover, the value per unit of the net product equals the direct and indirect labour cost per unit, *i.e.*, $P_a = w(l_aA' + l_bB' + l_cC')/S_a$. A similar rearrangement of the main system would show that the value of the total net product was equal to the value of the labour in the three industries.

10. The rearrangements, which essentially consist of adding down the columns of the systems instead of across the rows, reveal four results. First, they show how it is possible for (say) industries $b$ and $c$ in sub-system 1 to pay their wages bills (and profits if $r \neq 0$), even though in total, that is, from the point of view of the sub-system, their gross products are equal to the means of production in the current year. Secondly, they show that the values of

the components of the net product are not in general equal to the values added of the industries which produce them, even though, in total, the values of both are equal.[1] Thirdly, they show that the wages bill of any industry in the main system (say, industry *a*) is equal to the wages bill of the three industries in the sub-system in which the commodity is the net product, *i.e.*, $l_a Aw \equiv l_a A'w + l_b B'w + l_c C'w$. Fourthly—and this is the principal proposition—the relative values of the commodities (here interpreted as the net products, $S_a$, $S_b$ and $S_c$) are proportional to their direct and indirect labour costs *when the rate of profits is zero*. Sub-system 1 was chosen for illustrative purposes only, and since it also could be shown that the net products of sub-systems 2 and 3 are equal in value to their total wages bills, and therefore that their prices are equal to their direct and indirect labour costs per unit, this main proposition immediately follows.

11. To complete the analysis we show that these propositions still hold when not all the basic commodities exceed their use as means of production. Suppose that only two commodities are in surplus, even though there are three basics in the main system, that is, suppose that $S_c$ (say) $= 0$ in our example. Then we have two sub-systems, but since they are merely a rearrangement of the main system, all the labour and means of production will be shared between them, such that each has a net product equal to its counterpart in the main system. The national incomes will still equal the corresponding wages bills when $r = 0$; and so on. It may be asked: since *c* is not a component of the net product, what can be said of its value? This question can be answered in at least two ways. The sub-system is a conceptual notion, derived from, but not separate from, the main system. The same relationships therefore apply as much in it as in the main system. In particular, the relative prices and the rate of profits (if $r \neq 0$) which are the solutions of the equations implied in the main system are also the solutions of those in any sub-system. Therefore we may employ the concept of the sub-system to construct a self-replacing system which has a net product of one unit of *c*; or a set of self-replacing systems which have as their net products the amounts of the commodities which form the means of production in the *c* industry of the main system. Either of these constructions can then be used to show that when $r = 0$ the price of *c* is equal to its direct and indirect labour cost per unit. We conclude that the results derived from our three-basic-commodity example are quite general.

12. In conclusion, we note some further implications of the analysis. As we have seen, the total value added and the value of the net product of the system are equal, as both equal the incomes created by the year's production. However, the *commodities* associated with value added (value added does not consist of commodities but is the *value* of the difference between the value of gross output and the value of the means of production) are not in general the

[1] The equality of totals does not imply that (say)

$$(x_{aa}Ap_a + x_{ba}Bp_a + x_{ca}Cp_a) = (x_{aa}Ap_a + x_{ab}Ap_b + x_{ac}Ap_c)$$

same commodities as those which make up the net product of the system. This is clearly seen in a sub-system where, for example, the values of $B'$ and $C'$ exceed those of their corresponding means of production, even though $a$ is the only commodity in the net product. Amounts of commodities which are components of the net product can be calculated without the need to value commodities, but values added cannot be. Moreover, values added are associated with current production and the means of production employed in the year; while the net product is associated with current production and those parts of the gross product which are equal to the means of production.

13. The last point illustrates a general proposition, that the surpluses of commodities which may be of interest to the economy as a whole for, say, capital accumulation, in general differ in amount and kind from the surpluses which are of interest to businessmen, namely, values added, especially the profits contained therein. Businessmen are interested in values and incomes (values added), while governments and planners may be interested in real goods and services (net product). Finally, we may note that, once the technical conditions of production are given, the net product is determined independently of its division between wages and profits.

G. C. Harcourt

*University of Adelaide and Trinity Hall, Cambridge.*

Vincent G. Massaro

*Visiting N.A.T.O. Fellow, Cambridge.*

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# THE "LABOUR APPROACH" AND THE "COMMODITY APPROACH" IN MR. SRAFFA'S PRICE THEORY[1]

Both Sir Roy Harrod (Economic Journal, December 1961) and Mr. Collard (Economic Journal, March 1963) have suggested that of the two approaches in Mr. Sraffa's book,[2] the so-called "labour–cost approach" is inferior to the "commodity approach." The purpose of this Note is to show that if one uses Mr. Sraffa's device of "sub-systems"[3]—which unfortunately Mr. Sraffa elaborates less fully than the alternative device of "reduction to dated labour"—one gets round the difficulties which allegedly make the "labour–cost approach" inferior.

[1] I am indebted to Mr. Piero Sraffa for going through the first draft of this Note and for clarifying a point about the effect on rates of returns of conversion into sub-systems.

[2] The Production of Commodities by Means of Commodities, 1960.

[3] *Op. cit.*, Appendix A.