4 The law of supply and demand in the proof of existence of general competitive equilibrium

Carlo Benetti, Alejandro Nadal, and Carlos Salas

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Introduction

The proof of existence of a general competitive equilibrium is generally considered one of the most important and robust results of economic theory. The proofs of existence, which appeared in the 1950s, relied on results of topology, using a fixed-point theorem to demonstrate the existence of an equilibrium point. These proofs employ a suitable mapping, transforming points of a convenient set of prices and quantities onto itself. Our argument, in brief, is that the mappings used in these proofs are mathematically convenient but economically meaningless: they do not correspond to any plausible process of price variation.

To understand the mathematical strategy of the existence proofs, it may help to begin with a trivial example. In a one-commodity market, one would expect the change in price to reflect the excess demand for the commodity: price goes up when excess demand is positive, goes down when excess demand is negative, and remains unchanged at the market equilibrium point when excess demand is zero. In an n-commodity market, the mapping that determines price changes is more complex, but the underlying idea is similar: price changes are based on a function of prices and quantities, usually involving excess demand. An equilibrium is a vector of prices and quantities at which prices do not change because supply equals demand for all commodities – that is, a fixed point in the mapping which determines quantities and prices. If the sets and mappings have all the required topological properties, the mappings are guaranteed to have a fixed point, demonstrating the existence of general economic equilibrium.

The main objective of this chapter is to analyze the economic interpretations of the mappings involved in the proofs of existence. In the writings concerning the existence of equilibrium, the mathematical proof using a fixed-point theorem is accompanied by an economic interpretation of the relevant mappings. This interpretation evolved through time: in the 1950s it was considered that the mappings described a dynamic adjustment process, but later they were thought to express the law of supply and demand as a price variation rule without any reference to a dynamic

adjustment. This interpretation is commonly shared today in the relevant literature. The main finding of this chapter is that the second line of interpretation is as unacceptable as the first: in general, the mappings used in the proofs of existence contradict the price variation rule that is supposed to justify them from an economic standpoint.

If our analysis is correct, the single most important result of neoclassical theory in the past fifty years is a mathematical theorem devoid of any economic sense. Our results are a direct criticism of dominant economic theory from two points of view. The first pertains to the theoretical soundness and rigor of neoclassical theory. The second is more general, and concerns the relationship between mathematics and economic theory. These two aspects are of relevance today given that (i) on the basis of this purported logical coherence, neoclassical theory claims today to be the only available theoretical construct; and (ii) mathematization of economic theory is one of the most visible traits marking the evolution of the discipline during the past fifty years.[1] Our analysis relies on a thorough investigation of the mappings' behavior, something that surprisingly has attracted little or no attention since their appearance in the theoretical literature in the 1950s.

The first section describes the two economic interpretations of the mappings as they have evolved since the 1950s. In the second section we show that the three main mappings in the literature are inconsistent with the law of supply and demand.[2] In the third section we offer an explanation of the incompatibility between the mappings and the law of supply and demand. Our analysis leads to the question of whether a proof of existence of general equilibrium deprived of any economic meaning can be considered to be satisfactory. This important aspect of the problem is examined in our conclusion.[3]

Economic interpretations of the mappings used in the proof of existence of general equilibrium

The economic sense ascribed to the mappings used in the proofs of existence has evolved over time. In the first writings it was ascertained, sometimes implicitly, that the mappings described a dynamic price adjustment process leading to general equilibrium. However, as the first negative results concerning stability came to light, the economic justification of the mappings was modified and restricted to the law of supply and demand as a rule of price changes without reference to the effects of these price variations on excess demands in the following period. In the following subsection we examine these interpretations in more detail.

Interpretation of the proof of existence in terms of the dynamic adjustment process

The 1956 papers by Nikaido and Debreu respectively stressed the idea that the mappings used in the proof of existence of equilibrium were the mathematical expression of a dynamic adjustment process. Prices changed according to the law of supply and demand as a function of excess demand's signs, while excess demands, in turn, are modified according to the relation $\Delta z_{i,t+1}(p) = G_i(\Delta p_{i,t})$. If such a process converges towards a position of equilibrium, it is defined as stable.

This view was already held by Gale (1955), whose paper suggests a close relation between the proof of existence and the law of supply and demand, defined as the mechanism by which "prices eventually regulate themselves to values at which supply and demand exactly balance, these being the prices at economic equilibrium" (ibid.: 87). The most important texts that pursue this interpretation are the following.

Nikaido uses the following mapping

$$\theta_i(p) = \frac{p_i + \max(z_i, 0)}{1 + \Sigma_j \max(z_j, 0)} \qquad (i = 1, \ldots, n)$$

where p_i and z_i are the price and excess demand of commodity i respectively. The economic interpretation of $\theta(p)$ is advanced by Nikaido in the following terms:

> The mapping θ which appears in the proof of Theorem 16.6 may be interpreted as representing the behavior of the auctioneer who proposes a modification of prices responding to a nonequilibrium market situation.
>
> (Nikaido 1968: 268)

> Goods are exchanged in the market according to their prices. . . . If their demand and supply are not equal, current prices are induced to change under the influence of the "Invisible Hand". If new prices do not equate demand and supply, another round of price changes follows. Successive changes in prices with alterations in demand and supply continue until demand and supply are equated for all goods. In place of the Invisible Hand, we may suppose a fictitious auctioneer who declares prices p in the market. Participants in the market then cry out quantities they buy and sell. If their demand and supply do not match, the auctioneer declares a new set of prices p. $\theta(p, x)$ defined above may be interpreted as an adjustment mechanism of demand and supply that associates new prices with current prices and excess supply x.
>
> (Nikaido 1970: 321–22)

This interpretation was first put forward by Nikaido (1956). Consider a nonnegative price vector.

> If the corresponding total demand $X = \Sigma X_i$ does not match with the total available bundle A, the referee must try to set up a new price constellation which will be effective enough to let the individuals adjust their demands in such a way that the deviation of the total demand from A may be reduced. This scheme of the referee will be most effectively achieved by making the excess of the total monetary value PX to be paid by the individuals for X over their total available income PA as large as possible, i.e., by setting up a price constellation belonging to $\chi(X) = \{P \mid P(X-A) = \max Q(X-A) \text{ over all } Q \in S^k\}$. This function is multivalued and will be called the price manipulating function.
>
> (Nikaido 1956: 139)

At the time, Debreu (1956) was stating the same thing,[4] mainly that his mapping of Max ***p·z*** had "a simple economic interpretation: in order to reduce the excess demand, the weight of the price system is brought to bear on those commodities for which the excess demand is the greatest." He would later restate this as follows:

> [A]n increase in the price of a commodity increases, or leaves unchanged, the total supply of that commodity. This hints at a tendency for an increase in the price of a commodity to decrease the corresponding excess demand. It prompts one, when trying to reduce positive excess demand, to put the weight of the price system on those commodities for which the excess demand is the greatest.
>
> (Debreu 1959: 83)

> According to a commonly held view of the role of prices, a natural reaction of a price-setting agency to this disequilibrium situation [i.e., a price vector with nonzero excess demands] would be to select a new price vector so as to make the excess demand F(p) as expensive as possible.
>
> (Debreu 1974: 219)

According to Debreu (1982: 708), the economic interpretation of this mapping is quite clear, which may explain his allegiance to this mapping over the years:

> the maximization with respect to ***p*** of this [excess demand] function agrees with a commonly held view of the way in which prices perform their market-equilibrating role by making commodities with positive excess demand more expensive and commodities with negative excess demand less expensive, thereby increasing the value of excess demand.

Interpretation of the proof of existence in terms of the law of supply and demand

The previous interpretation found less support after the 1960s, especially after a paper by Scarf (1960). It became totally unacceptable in the 1970s after the negative results of Sonnenschein (1973), Mantel (1974), and Debreu (1974), who together resolutely demonstrated that the "commonly held view" on the "market-equilibrating role" of prices in the Arrow–Debreu model is utterly unjustified.

Explicit discussion of this interpretation is given by Hildenbrand and Kirman (1988: 106): "Even though an adjustement process may not converge, nevertheless *a fixed point p** of it exists." This is why

> If we confine ourselves to a fixed point of the adjustment process then this process, as such, has no real intrinsic economic content. We can then arbitrarily choose a process to suit our purpose. The only criterion is its mathematical convenience.

This does not mean that the mapping can remain economically meaningless, but that for its pertinence *in the proof of existence*, a price adjustment process does not have to be stable. The economic interpretation of the mappings in the proof of existence can be suitably based on the law of supply and demand, without any reference to a dynamic adjustment process.

This important point has not been completely grasped. A significant example can be found in the textbook by Mas-Colell *et al.* (1995). They use Debreu's correspondence and state (ibid.: 586), "*This makes economic sense*; thinking $f(.)$ as a rule that adjusts current prices in a direction that eliminates any excess demand, the correspondence $f(.)$ as defined above assigns the highest prices to the commodities that are most in excess demand" (emphasis added). Such interpretation of the mapping in terms of an implicit reference to the stability of equilibrium is surprising.

In contrast, after presenting mapping $\theta(p)$ (see p. 70), Varian (1992: 321) proposes a different interpretation: "This map has a *reasonable economic interpretation*: if there is an excess demand in some market, so that $z_i(p) > 0$, then the relative price of this good is increased" (emphasis added).

A straightforward assessment of this interpretation can be found in a book by Starr (1997: 101):

> We establish sufficient conditions so that excess demand is a continuous function of prices and fulfills the Weak Walras's Law. The rest of the proof involves *the mathematics of an economic story* [emphasis added]. Suppose the Walrasian auctioneer starts out with an arbitrary possible price vector (chosen at random, *crié au hasard*, in Walras's phrase) and then adjusts prices in response to the excess demand function $\boldsymbol{Z}(\boldsymbol{p})$. He raises the price of goods, k, in excess demand, $Z_k(\boldsymbol{p}) > 0$, and reduces the price of goods, k, in excess supply, $Z_k(\boldsymbol{p}) < 0$. He performs this price adjustment as a continuous function of excess demands and supplies while staying on the price simplex. Then the price adjustment function $\theta(\boldsymbol{p})$ is a continuous mapping from the price simplex into itself. From the Brouwer Fixed-Point Theorem, there is a fixed point $\boldsymbol{p}^0$ of the price adjustment function, so that $\theta(\boldsymbol{p}^0) = \boldsymbol{p}^0$.

And, furthermore: "The price adjustment function θ raises the relative price of goods in excess demand and reduces that of goods in excess supply while keeping the price vector on the simplex."

This statement leaves no doubt: the mapping used in the proof of existence is the expression of the law of supply and demand. The Walrasian auctioneer modifies prices according to the sign of excess demand, but the economic story is not affected by the effects of these price variations on excess demands.

Kreps's remarks on mapping $\theta(\boldsymbol{p})$ are as follows:

> Take the numerator first. We add to the old price p_k a positive amount *if* there is excess demand for good k at price $\boldsymbol{p}$. (This makes sense; raise the prices of goods for which there is too much demand). Then the denominator takes these new relative prices and rescales them so they sum to one again.
>
> (Kreps 1990: 212)

In the absence of further comments, the reader is left with the impression that, as the numerator, the mapping $\theta(\boldsymbol{p})$ makes economic "sense." This presentation is misleading, as we will see in the next section.

Mappings and the law of supply and demand

We will show that in the three most important mappings used in the proof of existence of a general competitive equilibrium, the price variation rule does not comply with the law of supply and demand, which is defined in the following subsection.[5] The mappings examined here are from Nikaido (1968, 1970, 1989), Arrow and Hahn (1971) and, finally, Arrow and Debreu (1954) and Debreu (1956, 1959).

The law of supply and demand

In the words of Arrow (1981: 141), the "familiar law of supply and demand" states that the price of any one commodity increases when the demand for that commodity exceeds the supply, and decreases in the opposite case. If we take strictly positive prices, these can be measured in terms of a *numéraire*.[6] We can also study prices expressed in terms of an abstract unit of account as elements of the n-dimension simplex $\boldsymbol{p} \in S_n \subset R^+_n$.

Let $\Delta p_i = p'_i/\Sigma P'_i - p_i/\Sigma P_i$, and let $z_i(\boldsymbol{p})$ denote the excess demand function for commodity i. The law of supply and demand prescribes a price variation such that

$$\Delta p_i = 0 \text{ if } z_i(\boldsymbol{p}) = 0, \text{ or if } z_i(\boldsymbol{p}) < 0 \text{ with } p_i = 0$$

$$\Delta p_i \cdot z_i(\boldsymbol{p}) > 0 \text{ in all other cases}$$

This is the price variation rule that lies behind the contemporary economic interpretation of the mappings used in the proof of existence. But as we show in the following subsection, the mappings do not respect this price variation rule.

Nikaido's mapping

Nikaido (1968, 1970, 1989) proves the existence of a general equilibrium by using the mapping already mentioned in the previous section:

$$\theta_i(p) = \frac{p_i + \max(z_i, 0)}{1 + \Sigma_j \max(z_j, 0)} \qquad (i = 1, \ldots, n)$$

where p_i and z_i are the price and the excess demand of commodity i respectively. The mapping transforms points in the unit simplex P_n into price vectors $\boldsymbol{p}$ contained in the unit simplex. Each element of the unit simplex P_n is a normalized vector of prices such that $\Sigma_i p_i = 1$. Homogeneity of degree 0 of the excess demand and supply functions in all prices allows the search for equilibrium price vectors to be limited to the unit simplex of R_n.

To determine whether mapping $\theta_i(\boldsymbol{p})$ satisfies the law of supply and demand, we will examine successively the following three cases: $z_i > 0$, $z_i < 0$, y $z_i = 0$.

Positive excess demand

In the case of $z_i > 0$, the law of supply and demand specifies an increase the price of commodity i. This implies $\theta_i(\boldsymbol{p}) > p_i$ and, in turn, according with mapping $\theta_i(\boldsymbol{p})$, this means that we must have

$$p_i + z_i > p_i \left[1 + \Sigma_j \max(z_j, 0)\right]$$

$$z_i > p_i \left[\Sigma_j \max(z_j, 0)\right]$$

$$z_i > p_i \cdot z_i + p_i \cdot \Sigma_{j \neq i} \max(z_j, 0)$$

In this case, because $p_i < 1$, then $z_i \cdot p_i < z_i$. The inequality is verified if for all other commodities $j \neq i$, excess demands are negative or null. If one commodity $j \neq i$ has a positive excess demand, then the condition may not be satisfied. Thus, $\theta_i(\boldsymbol{p})$ is not consistent with the law of supply and demand.

Negative excess demand

If $z_i < 0$, the price of commodity i must decrease: $\theta_i(p) < p_i$. Because max $(z_i, 0) = 0$, this inequality implies

$$p_i < p_i + p_i \cdot \Sigma_j \max(z_j, 0)$$

This condition is verified if there is at least one commodity $j \neq i$ with a positive excess demand, which is guaranteed by Walras's law. In this case, the price adjustment rule expressed by the mapping $\theta_i(\boldsymbol{p})$ appears to be the law of supply and demand. However, the price variation for good i depends not only on the sign of z_i, but also on the presence of positive excess demands for other goods, something not dictated by the law of supply and demand. Thus, if the mapping appears to be consistent with the law of supply and demand, it is by virtue of Walras's law.

Zero excess demand

When $z_i = 0$, the law of supply and demand ordains that price p_i must remain unchanged, thus $\theta_i(\boldsymbol{p}) = p_i$. But once again, we have problems to interpret mapping $\theta_i(\boldsymbol{p})$ as consistent with the law of supply and demand. What are the conditions under which this equality is verified? Because max $(z_i, 0) = 0$, we have

$$p_i = p_i + p_i \cdot \Sigma_j \max(z_j, 0)$$

This condition is verified if the second term in the right-hand side is zero, and this is the case when for all $j \neq i$, $z_j \leq 0$. Because of Walras's law, this is not possible except in general equilibrium. Outside of *general* equilibrium, there exists at least one commodity $j \neq i$ with positive excess demand. The price adjustment rule in mapping $\theta_i(\boldsymbol{p})$ carries with it the reduction of price p_i. This is in contradiction with the law of supply and demand.

The Arrow–Hahn mapping

For the ith component, the mapping used by Arrow and Hahn (1971) is

$$T_i(p) = \frac{p_i + \max(-p_i, z_i(p))}{1 + \Sigma_j \max(-p_j, z_j(p))}$$

Although it may be somewhat monotonous, an analysis similar to the previous one is required.

Positive excess demand

The price p_i must rise – that is, $T_i(\boldsymbol{p}) > p_i$. This can be expressed as follows:

$$z_i(\boldsymbol{p}) > p_i \cdot z_i(\boldsymbol{p}) + p_i \cdot \Sigma_{j \neq i} \max(-p_j, z_j(\boldsymbol{p}))$$

If there exists a commodity $j \neq i$ with a positive excess demand, the above condition is verified only if the value of $z_i(\boldsymbol{p})$ is sufficiently large to prevail over the positive value of $z_j(\boldsymbol{p})$. The price variation rule imposed by mapping $T(\boldsymbol{p})$ does not respect the law of supply and demand.

Negative excess demand

Price p_i must decrease – that is, $T_i(\boldsymbol{p}) < p_i$. Hence

$$p_i + \max(-p_i, z_i(\boldsymbol{p})) < p_i [1 + \Sigma_j \max(-p_j, z_j(\boldsymbol{p}))]$$

$$\max(-p_i, z_i(\boldsymbol{p})) < p_i \cdot \Sigma_j \max(-p_j, z_j(\boldsymbol{p}))$$

Obviously, the possibility of reducing the price of commodity *i* depends on the absolute values of p_i, $z_i(\boldsymbol{p})$, p_j and $z_j(\boldsymbol{p})$. Thus, the above inequality may not be verified. According to the values of these variables, we can obtain $T_i(\boldsymbol{p}) > p_i$; this means that, in spite of the excess supply for commodity *i*, the price imposed by $T_i(\boldsymbol{p})$ may increase.

Zero excess demand

When $z_i(\boldsymbol{p}) = 0$, we should have $T_i(\boldsymbol{p}) = p_i$. Thus

$$p_i + \max(-p_i, z_i(\boldsymbol{p})) = p_i [1 + \Sigma_j \max(-p_j, z_j(\boldsymbol{p}))]$$

$$p_i = p_i + p_i \cdot \Sigma_j \max(-p_j, z_j(\boldsymbol{p}))$$

Equality $T_i(\boldsymbol{p}) = p_i$ is verified only if $z_i(\boldsymbol{p}) = 0$ *and* if $z_j(\boldsymbol{p}) = 0$ for all commodities $j \neq i$. This is not what the law of supply and demand states.

Debreu's approach

Debreu (1959) considers a price vector $\boldsymbol{p}$ in the unit simplex $P_n = \{p \in R_n^+ \mid p \geq 0,\ \Sigma_i\ p_i = 1\}$, and the set of possible excess demands Z. He defines an aggregate excess demand correspondence $\zeta(p) = \xi(\boldsymbol{p}) - \eta(\boldsymbol{p}) - \{\omega\}$ (where $\xi(\boldsymbol{p})$ is the aggregate demand correspondence, $\eta(\boldsymbol{p})$ the aggregate supply correspondence and $\{\omega\}$ the vector of

initial endowments of the economy) which associates to each price vector $\boldsymbol{p} \in P_n$ a vector $\boldsymbol{z} \in Z$. A new correspondence $\mu(\boldsymbol{z})$ then associates to $\boldsymbol{z}$ a vector of prices within P_n such that $\boldsymbol{p} \cdot \boldsymbol{z}$ is maximized:

$$\mu(\boldsymbol{z}) = \{p \in P_n \mid \boldsymbol{p} \cdot \boldsymbol{z} = \max P \cdot \boldsymbol{z}\}$$

Debreu then defines a new correspondence ψ of set $P_n \times Z$ on itself $\psi(\boldsymbol{p}, \boldsymbol{z}) = \mu(\boldsymbol{z}) \times \zeta(\boldsymbol{p})$. This mapping $\psi(\boldsymbol{z}, \boldsymbol{p})$ implies that to each vector $\boldsymbol{z}$ a price vector $\boldsymbol{p}$ is associated in order to maximize $\boldsymbol{p} \cdot \boldsymbol{z}$. This is what Debreu (1959: 83) calls "the central idea in the proof," which is then described in the following terms: "Let H be the set of commodities for which the component of $\boldsymbol{z}$ is the greatest. Maximizing $\boldsymbol{p} \cdot \boldsymbol{z}$ on P_n amounts to taking $\boldsymbol{p} \geq 0$ such that $p_h = 0$ if $h \notin H$, and $\Sigma_{h \in H} p_h = 1$."

The price adjustment rule is the following: the commodity k with the highest excess demand in vector $\boldsymbol{z}$ is chosen, such that $z_k \geq z_i$, $\forall\ z_i \in Z$, $i \neq h$. The new price vector resulting from correspondence $\mu(\boldsymbol{p})$ has all of its components $p_{i \neq k} = 0$ and component $p_k = 1$ (because no linear combination of the price vector and the excess demand vector results in a higher value than $p_k \cdot z_k$). That is to say, outside of the fixed point, the prices of commodities with positive excess demands (at positive prices) inferior to the largest excess demand are reduced to zero. Their prices are brought to zero for the simple reason that their excess demand is not superior to the other excess demands.

An alternative approach to examine this is as follows. Let $\boldsymbol{p}$ be a price vector, $\boldsymbol{z}$ the vector of excess demands calculated at these prices and $\boldsymbol{p}'$ the new price vector resulting from the law of supply and demand. Necessarily we have $\boldsymbol{p}' \cdot \boldsymbol{z} > \boldsymbol{p} \cdot \boldsymbol{z}$. The consequence of this law is that, outside the fixed point, the aggregate value of excess demand must increase. But the economic meaning of this result stems from the same reason advanced by Debreu: the increase (or decrease) of the prices of commodities with positive (or negative) excess demand. Thus, contrary to Debreu's assertion, the value of $\boldsymbol{p} \cdot \boldsymbol{z}$ cannot be a maximum without contradicting the law of supply and demand. This is self-evident: to reach this maximum, the prices of commodities with excess demands that are both positive and inferior to the largest must be reduced to zero; in the case that several commodities have the same largest excess demand, all of their prices, except one, can be reduced to zero, reserving $p = 1$ for the exception.[7] There is here a brazen contradiction with the law of supply and demand.[8]

These considerations should help explain Arrow's reservations: "this rule is somewhat artificial" (1972: 219) and, later, Debreu's (1989: 134):

> Maximizing the function $\boldsymbol{p} \to \boldsymbol{p} \cdot \boldsymbol{z}$ over $\mathrm{P_n}$ carries to one extreme the idea that the price-setter should choose high prices for the commodities that are in excess demand, and low prices for the commodities that are in excess supply.

But these calls for caution are useless: the mapping that maximizes $\boldsymbol{p}\cdot\boldsymbol{z}$ is *totally* artificial, and it does not carry to one extreme the law of supply and demand, but utterly *contradicts* it.[9]

The special case of a two-commodity economy

Consider a two-commodity economy with p_1, p_2 and z_1, z_2, the prices and excess demands of commodities 1 and 2 respectively, and suppose that all customary conditions for the existence of equilibrium are verified. By virtue of Walras's law, $\boldsymbol{p}\cdot\boldsymbol{z}=0$, and thus $z_1\cdot z_2<0$.

Consider Nikaido's correspondence:

$$\theta_i(p)=\frac{p_i+\max(z_i,0)}{1+\Sigma_j\max(z_j,0)}$$

when $z_1>0$. Because $z_2<0$, $\theta_1(p)>p_1$ is true if $z_1>p_1z_1$, the last inequality holds since $p_1<1$. If $z_1<0$, we have $p_1z_2>0$, which is equivalent to $\theta_i(\boldsymbol{p})<p_1$. Since these inequalities are verified, the price of commodity 1 increases in the first case and decreases in the second.

We arrive at the same conclusion considering the correspondence of Arrow–Hahn:

$$T_i(p)=\frac{p_i+\max(-p_i,z_i(p))}{1+\Sigma_j\max(-p_j,z_j(p))}$$

Suppose $z_1>0$. Because $p_1<1$, we have $(1-p_1)z_1>0$. Since $z_2<0$, $p_1[(\max(-p_2,z_2)]<0$, thus $(1-p_1)z_1>p_1[(\max(-p_2,z_2)]$. The conditions for increasing p_1 are satisfied.

Consider now $z_1<0$. Let $u_1=\max(-p_1,z_1)$. Then $(1-p_1)u_1<0$, $z_2>0$ and $\max(-p_2,z_2)=z_2$. Therefore, $p_1(\max(-p_2,z_2))>0$ and $(1-p_1)u_1<p_1(\max(-p_2,z_2)$. Thus the conditions for the reduction of p_1 are verified.

Finally, the price adjustment rule imposed by Debreu mapping which maximizes the value of $\boldsymbol{p}\cdot\boldsymbol{z}$ yields the following result. If $z_1>0$, we have $z_2<0$ and p_1 is increased until it equals 1. If $z_1<0$, p_1 is reduced until it becomes 0. In the special case of a two-commodity economy, the property $\Delta p_i\cdot z_i(\boldsymbol{p})>0$ is verified by virtue of Walras's law, and *not* by the law of supply and demand.

Synthesis of results

1 $z_i>0$
  (a) $z_i>0\Rightarrow \mathrm{p_i}$ increases
  (b) p_i increases $\Rightarrow \mathrm{z_i}>0$

For correspondences $\theta_i(\boldsymbol{p})$ and $T_i(\boldsymbol{p})$, statement (a) is false and (b) is true. Therefore, $z_i>0$ is the necessary condition, but not sufficient, for the increment in p_i.

2 $z_i<0$
  (a) $z_i<0\Rightarrow p_i$ decreases
  (b) p_i decreases $\Rightarrow z_i<0$

For correspondence $\theta_i(\boldsymbol{p})$, statement (a) is true by virtue of Walras's law, but statement (b) is false. Thus, $z_i<0$ is the sufficient condition, but not the necessary condition for the reduction of p_i.

For correspondence $T_i(\boldsymbol{p})$, both statements are false: $z_i<0$ is neither the sufficient nor the necessary condition for the reduction of p_i.

3 $z_i=0$
  $z_i=0\Rightarrow \mathrm{p_i}=\theta_i(\boldsymbol{p})$
  $\mathrm{p_i}=\theta_i(p)\Rightarrow z_i=0$

For correspondence $\theta_i(\boldsymbol{p})$, (a) is false, but (b) is true only if $z_j=0$ for all $j\neq i$. Thus, we have that $z_i=0$ is a sufficient but not a necessary condition for $p_i=0$.

For correspondence $T_i(\boldsymbol{p})$, (a) and (b) are both false. Thus $z_i=0$ is neither the necessary nor the sufficient condition for $T_i(\boldsymbol{p})=p_i$.[10]

The law of supply and demand and the normalization of prices

The nature of the problem occupying our attention is clearly revealed if we follow the different stages of the construction of the mappings as exemplified in Arrow and Hahn's (1971: 25–27) procedure. The starting point is a two-commodity economy for which four price-variation rules, valid also in the general case of an *n*-commodity economy, are adopted:

(i) Raise the price of the good in positive excess demand.
(ii) Lower or at least do not raise the price of the good in excess supply, but never lower the price below zero.
(iii) Do not change the price of a good in zero excess demand.
(iv) Multiply the resulting price vector by a scalar, leaving relative prices unchanged, so that the new price vector you obtain is in S_n.

(Arrow and Hahn 1971: 25–27)

In the construction of the correspondence, we first seek for a continuous function $M_i(\mathbf{p})$ with the following three properties:

(1) $M_i(\boldsymbol{p}) > 0$ if and only if $z_i(\boldsymbol{p}) > 0$
(2) $M_i(\boldsymbol{p}) = 0$ if $z_i(\boldsymbol{p}) = 0$
(3) $p_i + M_i(\boldsymbol{p}) \geq 0$

It is intended that $M_i(\boldsymbol{p})$ represent an adjustment to an existing price so that a price vector $\boldsymbol{p}$ is transformed into a new price vector with components $p_i + M_i(\boldsymbol{p})$.

(Arrow and Hahn 1971: 25–27)

There are correspondences with properties P1–P3, for example:

$M_i(\boldsymbol{p}) = \max\,(-p_i, k_i \cdot z_i(\boldsymbol{p}))$, where $k_i > 0$.

[I]f we interpret $(p_i + M_i(\boldsymbol{p}))$ as the *i*th component of the new price vector that the mapping produces, given $\boldsymbol{p}$, the procedure for finding these new prices satisfies the rules discussed earlier. However, while all $(p_i + M_i(\boldsymbol{p}))$ are certainly non-negative, there is nothing to ensure that they will add up to one. In other words, . . . there is no reason to suppose that $(\boldsymbol{p} + \boldsymbol{M}(\boldsymbol{p}))$ is in S_n when $\boldsymbol{p}$ is in S_n. Since we seek a mapping of S_n into itself, we must modify the mapping.

(Arrow and Hahn 1971: 25–27)

This is where the price normalization implied by rule (iv) intervenes and the result is correspondence

$$\boldsymbol{T}(\boldsymbol{p}) = \frac{\boldsymbol{p} + \boldsymbol{M}(\boldsymbol{p})}{[\boldsymbol{p} + \boldsymbol{M}(\boldsymbol{p})]\boldsymbol{e}}$$

According to Arrow and Hahn, this is an "obvious way" of solving the difficulty they identified (see also Arrow 1968: 117). But this assertion is incorrect, because rule (4) modifies the initial mapping so as to make it *noncompliant with the first three rules*.

Our analysis of the most important mappings used in the proof of existence of GCE (pp. 74–77) reveals that under these conditions, the adjustment of price p_i does not depend so much on the sign of $z_i(\boldsymbol{p})$ as on the relation between $z_i(\boldsymbol{p})$ and the other $z_j(\boldsymbol{p})$ for $j \neq i$. It is the relative weight of $z_i(\boldsymbol{p})$ within the set of excess demands that has an influence on the direction of the change in p_i. This is the source of the strange price adjustment mechanism established by these correspondences: in a market i with positive excess demand, the price can increase or decrease depending on the relative importance of the excess demands on the other markets.[11] The interdependencies acting on the direction of the price variation of the mappings are a direct consequence of the normalization of the price system.

The predicament can be stated as follows. *In order to avoid falling outside of the price simplex, one leaves the law of supply and demand*: we either have a fixed point and the mapping is devoid of economic sense; or we use a correspondence with an economic meaning, but lose the fixed point.[12]

Conclusion

We can now summarize our key findings. The proofs of existence for a general competitive equilibrium are associated with an economic interpretation of the mappings used in the demonstration. We have shown that the interpretation of price variation generated by these mappings in terms of the law of supply and demand cannot be accepted.[13] With greater strength, this conclusion can be applied to interpretations in terms of a dynamic adjustment process.

The point is not a defense or critique of the law of supply and demand as it is conceived and presented in the framework of general equilibrium theory. What we are simply stating is that, first, this definition is unanimously accepted. Second, the authors we consider here claim that the mappings used in their proof of existence of equilibrium obey this law. Third, our analysis reveals that this is not the case. As a consequence, there is a difficulty in the proof of existence insofar as that which is actually accomplished does not correspond with what is claimed to be achieved.

It could be thought that because an "abstract economy" intervenes in the proof of existence, there is no need to provide an economic interpretation of the mappings. In point of fact, the economic interpretation of the mappings is described and justified precisely as the concept of an abstract economy is introduced by Arrow and Debreu. In their first proof of existence, advanced in 1954, which relies on the construction of an abstract economy, these authors propose an economic interpretation of their mapping precisely in terms of the law of supply and demand. The insistence on resorting to economically meaningful mappings is present in all of the relevant works of Arrow (including his conference on the occasion of the Nobel Prize), Debreu and Hahn. Debreu himself advances as the central justification of his excess demand approach the fact that it has a clear and simple economic interpretation.[14]

These authors' approach is quite correct, for the abstract economy they build is not isolated from the original economy, and the fundamental laws of the latter apply to the former. Or to put it in other terms, it is inconceivable that the rules that apply in the abstract economy contradict the laws of the original economy. The fact that we can deal with an "abstract" economy does not eliminate the fact that we are dealing with an "economy" subject to economic "laws". This is precisely the reason why it is possible to make the "return trip" from the abstract to the original economy in the attempt to complete the proof of existence of equilibrium. Thus, the construction of an "abstract economy" in no way justifies the idea that the mappings can be exempt of an economic interpretation.[15]

We thus arrive at the following crossroads. If it is considered that only the mathematical properties of the mappings are necessary, quite independently of their economic meaning, it is difficult to understand why claims to the contrary are so abundant. If the mappings are considered to have an economic meaning, as it is ascertained, then the use of mappings that lack such an economic meaning entails the lack of pertinence of the proof of existence from the economic viewpoint, whatever the mathematical properties of the intervening sets and mappings. Clarifying this situation is important because, due to the shortcomings of stability theory, the existence theorems play an all-important role in economic theory.

From our standpoint, we consider that if, mathematically, an economic equilibrium can be represented as a fixed point of a suitable mapping, it does not follow that every fixed point is an economic equilibrium. This depends on the nature of the intervening variables and the definition of the mapping used in the proof of existence of equilibrium. Given the nature of the task at hand, the rest point determined by the fixed-point theorem must be an economic rest point representing a state of the economy in which economic forces intervening in price formation are in balance. The search for a mapping with an economic meaning is thus a legitimate concern. It would be rather surprising to use a mapping that did not represent the law of supply and demand to demonstrate, by means of its fixed point, the existence of an equilibrium between supply and demand.

In the mappings used, the excess demand z_i generates a variation of price p_i that contradicts the law of supply and demand. This is true regardless of the sign of excess demand (positive or negative), as well as when excess demand is zero. If, in the fixed point, no individual prices change, this is not by virtue of the law of supply and demand: price p_i does not change only when $z_i = 0$ *and* $z_j = 0$, for all $j \neq i$. The excess demand $z_i = 0$ is a necessary condition for keeping p_i unchanged, but it is not a sufficient condition, contrary to what is stated by the law of supply and demand. Thus, whichever point over the mappings' domains is considered, such mappings are deprived of the economic meaning commonly attributed to them.

We reject the idea that *only* the mathematical properties of the proof should be taken into account. We have not encountered this proposition under the penmanship of the founders of contemporary general equilibrium theory, nor in later presentations. On the contrary, as we have seen, the authors have explicitly described the economic interpretation that they claim is inherent to the mappings they use. The task now is to draw out the consequences of the fact that, since the said mappings do not have the meaning attributed to them, the main result of the modern neoclassical theory is a mathematical theorem devoid of economic sense.

Notes

1 A recent, and lively, discussion of the relation between economic theory and mathematics can be found in d'Autume and Cartelier (1997).

2 We do not examine the proofs of existence that rely on the results of welfare theory (Arrow and Hahn 1971), nor do we consider the existence results that rely on assumptions of differentiability of individual supply and demand functions. It is true that in the context of general equilibrium theory, global analysis represents an approach that is closer to the older traditions (Smale 1989). Nonetheless, the crucial point for our purposes is that work along these lines (Smale 1981; Mas-Collel 1985) imposes assumptions that are more restrictive than those required by Arrow–Debreu models. Thus, our chapter is concerned with proofs of existence of general equilibrium in the more general setting.

3 We assume the reader is familiar with the techniques used in the proof of existence of general competitive equilibrium.

4 As to Debreu's approach, Hildenbrand (1983: 20) describes it as follows: "Debreu used another method of proof in his further work on competitive equilibrium analysis . . . i.e. the 'excess demand approach' because he thought that this method of proving existence is more in line of traditional economic thinking."

5 This carries negative implications for the two economic interpretations described in the previous section, for the economic interpretation based on a dynamic price adjustment process rests on the assumption that the law of supply and demand is respected by the mappings.

6 We are not concerned here by the effects of the choice of *numéraire* on stability.

7 "[T]otal prices must add up to one, but this total is to be distributed only over those commodities with maximum excess demand" (Arrow 1972: 219). The mapping used by Arrow and Debreu (1954) and Debreu (1959) finds its origins in the hypotheses of the maximum theorem. According to Takayama (1988: 254), although Debreu used the maximum theorem in his *Theory of Value* (1959) in order to establish the upper semicontinuity of the demand and supply functions, no explicit mention of the literature on the theorem (in particular, the seminal work of C. Berge) was made by him. Debreu (1982) does make an explicit reference to Berge's maximum theorem. This theorem can be used to prove the upper semicontinuity of multivalued correspondences, and is employed to establish this property for the supply and demand correspondences. Although the correspondence max $\boldsymbol{p} \times \boldsymbol{z}$ does exhibit this property, the difficulty is that in order to ensure the property of upper semicontinuity, the proof relies on a correspondence lacking a reasonable economic meaning. The predicament here is that the property of upper semicontinuity is guaranteed at the cost of rendering the correspondence incompatible with the law of supply and demand.

8 In Arrow and Debreu (1954: 275), a "market participant" with a price-setting role is introduced. This agent, renamed by Debreu (1982: 134) the "fictitious price-setting agent" and endowed with a "utility function" that "is specified to be $\boldsymbol{p} \times \boldsymbol{z}$", chooses a price vector $\boldsymbol{p}$ in P for a given $\boldsymbol{z}$ and "receives $\boldsymbol{p} \times \boldsymbol{z}$". As we have seen, this new price vector $\boldsymbol{p}$ maximizes $\boldsymbol{p} \times \boldsymbol{z}$, which implies, outside the fixed point, that all prices are zero except the price of the commodity with the largest excess demand. Arrow and Debreu (1954: 274–75) continue: "Suppose the market participant does not maximize instantaneously but, taking other participants' choices as given, adjusts his choice of prices so as to increase his pay-off. For given $\boldsymbol{z}$, $\boldsymbol{pz}$ is a linear function of $\boldsymbol{p}$; it can be increased by increasing p_h for those commodities for which $z_h > 0$, decreasing $z_h < 0$ (provided p_h is

not already zero). But this is precisely the classical 'law of supply and demand', and so the motivation of the market participant corresponds to one of the elements of the competitive equilibrium." This behavior, which is totally artificial, reinforces our conclusion. Instead of abruptly contradicting the law of supply and demand, the contradiction is obtained gradually. In this case, the law holds as long as the market participant does not maximize his utility function, and ceases to hold when this agent at last behaves according to the rationality that is assigned to him.

9 Nikaido (1968: 267) also presents this type of correspondence as an alternative way to approach the proof of existence of a competitive equilibrium. Correspondence η yields equilibrium solutions for the excess-supply correspondence χ as fixed points of mapping:

$$f(u,p) = \chi(p) \times \eta(u) \colon \Gamma \times P_n \to 2^{\Gamma \times Pn}$$

where $\boldsymbol{u}$ represents the vector of excess supplies, and $\eta(\boldsymbol{u}) = \{\mathrm{r} \mid \text{minimizes } \boldsymbol{u}\cdot\boldsymbol{q} \text{ for all } \boldsymbol{q} \in P_n\}$. Our remarks on the Arrow–Debreu mapping apply *mutatis mutandis* to this approach to the proof of existence of a general competitive equilibrium.

10 If we consider relative prices of the form p_i/p_j, then

(a) $z_i > 0$ and $z_j < 0$ then p_i/p_j increases;
(b) p_i/p_j increases, then $z_i > 0$ and $z_j < 0$

Whichever correspondence is considered, $\theta_i(\boldsymbol{p})$ or $T(\boldsymbol{p})$, (a) is true and (b) is false. Thus, $z_i > 0$ and $z_j < 0$ is the sufficient condition, but not the necessary condition for the increase of p_i/p_j. The same conclusion applies in the opposite case ($z_i < 0$ and $z_j > 0$). Evidently, the comparison of "relative prices" does not furnish indications about the state of supplies and demands which, through these correspondences, have generated the price variation. The only thing it reveals is that if, for example, $\theta_i(\boldsymbol{p})/\theta_j(\boldsymbol{p}) > p_i/p_j$, then $z_i > z_j$. But these excess demands can be both positive or both negative.

11 Note that this rule which brings to bear the relative weight of excess demands in the other markets on the direction of price variations in one market has nothing to do with the type of interdependencies commonly considered in general equilibrium theory, such as substitution and income effects. The latter concern the effects of the changes in the prices on the excess demands and not the effects of changes in excess demands on prices. None of these interdependencies can explain why the price of one commodity decreases (increases) when its excess demand is positive (negative).

12 Would it be possible to avoid this predicament? This would imply seeking a fixed point in a correspondence consistent with the law of supply and demand, for example $p_i + M_i(\boldsymbol{p})$. To our knowledge this has not been attempted. The reason for this probably lies in the additional restrictions that would have to be imposed on the supply and demand correspondences. As is well known from the work of Sonnenschein, Mantel, and Debreu, there is no economic justification for such restrictions. Moreover, such additional constraints on these correspondences would limit the generality that is commonly attributed to the proof of existence in Arrow–Debreu models.

13 It is straightforward to construct numerical examples in which the relevant assumptions hold (Walras's law and prices belong to the unit simplex) but where price changes contradict the law of supply and demand.

14 In their classic 1954 paper, Arrow and Debreu set the precedent as their concept of an abstract economy includes the market participant, his payoff function (max $\boldsymbol{p} \times \boldsymbol{z}$) and the economic behavior of consumers and producers. Debreu's survey article (1982: 708) is quite explicit on this point, for in order to

cast the abstract economy "in the form of the general model of a social system," Debreu introduces a fictitious market agent whose role is to choose a price vector $\boldsymbol{p} \in P$ and whose utility function depends on choosing $\boldsymbol{p}$ so as to make excess demand as expensive as possible.

15 The construction of an abstract economy implies, among other things, modifying the original possibility sets of individual producers and consumers in order to ensure boundedness. This property is in turn required to ensure that individual supply and demand functions are defined. Chapter 2 by Nadal, on the building blocks of general equilibrium theory, examines the shortcomings of this procedure.

Index